Generalised and non-invertible symmetries: a short survival journey

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1 Introduction

Symmetries are the basic concept of theoretical physics since 1900. They are the basic object we use to characterize a system: Noether theorems in classical mechanics, angular momentum in quantum mechanics, selection rules in atomic and molecular physics, Bravais lattice in condensed matter physics, the structure of special and general relativity, the fundamental interactions and the mass of elementary particles, the structure of Quantum Field Theories (QFTs) and their conformal and supersymmetric extensions, the holographic map, string theory compactifications; these are only few of the fields where symmetries play a crucial role. The idea is to generalize symmetries to something which may be more powerful in constraining physical systems. The basic motivetion emerge from the question: "There are more complicated objects that are charged under global symmetries rather than points (i.e. particles)?" and the answer open a Pandora's box with inside a lot of new ideas on what symmetries are and how can be interpreted. All it is started from a pioneering paper of Gaiotto, Kapustin, Seiberg and Willett where a the fundamental idea to view symmetries as topological defects was born [1]. However the term topological here is misleading as we will see later. The new vision open the doors to the generalisation to higher form symmetries where topological defects associated have codimension grater than one and the charged object are not point-like but they have dimension grater than one according to the form degree.

Very fast new ideas come out, for example the higher group symmetries, which mimic the non-abelian structure of the 0-form symmetry admitting different form symmetries in the same theories and looking at the possible mixings can occur when both higher and 0-form symmetries are present. Another very fruitful idea is that of non-invertible symmetries where the group law of composition of different transformations is not longer valid anymore but it is supplanted by something more complicated.

All these new kind of symmetries are useful to constrain the dynamic of the theories thanks to their Ward identities and are important in the context of duality, where several different Lagrangians describe the same theory. In such a situation the gauge symmetries of the dual descriptions do not have to match. But the global symmetries must match and we believe that the same is true for higher form global symmetries. The various dual descriptions should have the same such symmetries and the charged operators in the dual descriptions should also match. A lot of explicit constructions are available; however, most of these constructions are very technical and often pass thought some gauging of other symmetries. Moreover, these higher form symmetries, can be spontaneously broken and all the gauge field can be interpreted as Goldstone bosons of some suitable higher form symmetry. Therefore gauge symmetry is not the only way to constrain a bosonic field to be massless. Despite this interpretation is very interesting, we have to keep in mind that is commonly accepted the idea, supported by theoretical evidences[1], that in a full theory of quantum gravity no global symmetries exist; however, generalised symmetries can be used to show interesting results in the context of swampland conjectures [2]. Moreover some example of non-invertible symmetries can be found in the context of the AdS$\times T^{1,1}$ holographic correspondence [3] and would be interesting to study what happens in more general Sasaki-Einstein manifold cases and try to characterize them form the geometry of the associated Calabi-Yau cone.

Obviously, to a generalized symmetry we would associate a generalized charge. This is the case and the associated charge is called a q-charge for a q-form symmetry [4]. Authors argued that q-charges of a standard global symmetry, also known as a 0-form symmetry, correspond to the so-called $(q+1)$-representations of the 0-form symmetry group, which are natural higher- categorical generalizations of the standard notion of

[1]These evidences came form different approaches to quantum gravity such as the loop quantum gravity, the stringy or holographic point of view.

representations of a group. This generalizes already our understanding of possible charges under a 0-form symmetry. Just like local operators form representations of the 0-form symmetry group, higher-dimensional extended operators form higher-representations. This statement has a straightforward generalization to other invertible symmetries: q-charges of higher-form and higher-group symmetries are $(q+1)$-representations of the corresponding higher-groups.

In what follows we first talk about higher form symmetries discussing about their generalities; than we are going to move in the direction of higher group symmetries and in the and to the non-invertible case.

2 Generalised symmetries

The main idea is effectively simple: the charged objects are not particles, but higher dimensional branes and the charged observables are not zero-dimensional local operators, but higher dimensional objects. It is important to stress that this generalised global symmetries can be afflicted by 't Hooft anomalies, meaning that there is an obstruction to the gauging of these global symmetries. In other words, the global symmetry if free from anomalies but if we couple the theory to a background gauge field the symmetry became anomalous. An example of this is quantum chromodynamics with N_f massless fermions: This is a $SU(N_c)$ gauge theory with N_f massless Dirac fermions and has the global flavor symmetry $SU(N_f)_L \times SU(N_f)_R \times U(1)_V$. This global symmetry suffer of a 't Hooft anomaly.

In order to avoid confusion we should clarify the terminology used in literature. First, for extended observables i.e lines, surfaces, etc, we will use the words operator and defect interchangeably. When the extended observable is placed at a given time, it can be interpreted as an operator acting on the Hilbert space; otherwise, when it is stretched along the time direction, it is not an operator in the theory, but a defect so it describes the same theory with a different Hilbert space. In a full euclidean point of view we can use the terms operator/defect/observables interchangeably. Choose a Euclidean quantum field theory; this means, grab a collection of dynamical fields Φ that you will path-integrate over and a collection of background fields (or background data) $\mathcal{B}$, that will be your choice and won't be dynamical. A p-dimensional defect, is a p-dimensional operator made out of background fields: $D_p(\mathcal{B})$. This defects can be genuine or non-genuine. A genuine defect $D_p(\mathcal{B}, \Sigma_p)$ of dimension p is a defect operator that can be inserted along any dimension p sub-manifold Σ_p of the p-dimensional manifold M_d. A non-genuine defect is a defect that is not genuine. In the following figure defects D_1' and D_2 are genuine while defects D_1 and D_0 are non-genuine. In the following we will consider only genuine defects unless otherwise specified.

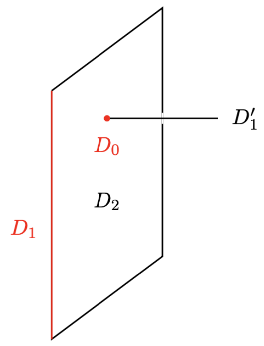

Figure 1: Different genuine and non-genuine defects

It is fundamental to underline that the presentation of generalised global symmetries will not rely on an underlying Lagrangian. Indeed, we will characterize the charges and the charged objects as abstract operators. This is an intrinsic description of the symmetry, which holds even when there is no Lagrangian description of the theory, such the case of the $6d$ $(2,0)$ superconformal theory, or when there is more than one Lagrangian, such the case of dual theories.

2.1 Ordinary symmetry from a generalised prospective

As well known, symmetry transformations form a group. This means that the composition of a symmetry transformation is again a symmetry transformation and:

1. composing two symmetry transformation in different ways give rise to the same final symmetry trasformation;
2. there exist the trivial transformation;
3. for every symmetry transformation there exist the inverse transformation and this is again a symmetry of the theory.

In a more precise way, given a set of element G this has the group structure if there exist a binary product $\cdot : G \times G \to G$ such that

1. for all $g, g' \in G$ we have $g \cdot g' = g'' \in G$ (if G abelian $g \cdot g' = g' \cdot g$);
2. exist e such that if $g \in G$ then $e \cdot g = g \cdot e = g \in G$ (left or right neutral element is sufficient);
3. for every $g \in G$ there exist $g^{-1} \in G$ such that $g^{-1} \cdot g = e$ (left or right inverse is sufficient).

It is useful to consider the symmetry transformation as an operator associated with the riemannian[2] manifold $(\Sigma_{(d-1)}, \langle \bullet, \bullet \rangle) \subset (M_d, \langle\langle \bullet, \bullet \rangle\rangle)$, where $(M_d, \langle\langle \bullet, \bullet \rangle\rangle)$ is a

[2]This means a manifold M is endowed by sign-definite non-degenerate symmetric bilinear rank 2-tensor field $\langle \bullet, \bullet \rangle \in \Gamma(SM)$ such that $\langle \bullet, \bullet \rangle : \chi(M) \times \chi(M) \to \mathbb{R}$.

pseudo-riemannian[3] manifold and $\langle \bullet, \bullet \rangle = \langle\langle \bullet, \bullet \rangle\rangle\big|_{T\Sigma_{d-1}}$, $D^g_{d-1}(\mathcal{B}, \Sigma_{(d-1)})$ with $g \in G$ where G is global symmetry group and we are considering a d-dimensional theory. The fact that $D^g_{d-1}(\mathcal{B}, \Sigma_{(d-1)})$ is associated with a symmetry means that the dependence on $\Sigma_{(d-1)}$ is topological, meaning that it is unchanged when $\Sigma_{(d-1)}$ is deformed slightly; it can change only when the deformation of $\Sigma_{(d-1)}$ crosses a point-like local operator $\mathcal{O}^{(0)}$, with domain on a 0-dimensional manifold, charged under the symmetry we are considering. However, the term is used in an improper way; indeed what we are saying is that we have the equality of correlation functions

$$\langle ...D^g_{d-1}(\mathcal{B}, \Sigma_{(d-1)})...\rangle = \langle ...D^g_{d-1}(\mathcal{B}, \Sigma'_{(d-1)})...\rangle \tag{2.1}$$

where $\langle ...D^g_{d-1}(\mathcal{B}, \Sigma_{(d-1)})...\rangle$ denotes the correlation function obtained by changing the locus of $D^g_{d-1}(\mathcal{B}, \bullet)$ from $\Sigma_{(d-1)}$ to $\Sigma'_{(d-1)}$ by a homotopy that does not intersect the loci of other defects participating in the correlation function, and the loci of other defects are not changed. Before moving on let us explain what an homotopy is. A homotopy between two continuous functions $f(x)$ and $g(x)$ from a topological space X to a topological space Y is defined to be a continuous function $H : X \times [0, 1] \to Y$ such that

$$H(x, 0) = f(x), \quad H(x, 1) = g(x) \qquad x \in X. \tag{2.2}$$

This means that we can continuously deform the two functions one in the other. In our case this is applied to Σ_{d-1} and $\Sigma'_{(d-1)}$ seen as functions from X to itself and equality 2.1 means invariant under homotopy. The invariance under homotopy has not to be confused with the topological property; in fact an object is said to be topological invariant if it is invariant under homeomorphisms, i.e, bijective continuous functions with continuous inverse. Since all homeomorfisms are homotopies but not all homotopies are homeomorphisms (a homotopy has not to be invertible) the property to be homotopy invariant is less strong to the property to be topological invariant. Therefore these defects should not be called "topological" but "homotopical". Examples of topological and homotopical invariant quantities are, respectively, the Euler characteristic and the fundamental group. Equality 2.1 can be rephased using homotopy groups: the correlator $\langle ...D^g_{d-1}(\mathcal{B}, \Sigma_{(d-1)})...\rangle$ does not depend from the choice of the representative in $\pi_{d-1}(M_d)$.

In the continuous case $D^g_{d-1}(\mathcal{B}, \Sigma_{(d-1)})$ can be obtained by exponentiating the charge $Q(\Sigma_{(d-1)})$ constructed from the Noether 1-form closed current $j^{(1)}$

$$Q(\Sigma_{(d-1)}) = \int_{\Sigma_{(d-1)}} \star j^{(1)}. \tag{2.3}$$

where $\star : \Omega^k(M_d) \to \Omega^{d-k}(M_d)$ is the unique linear function between k-forms and $(d-k)$-forms such that

$$\alpha \wedge \star\beta = (\alpha, \beta)\omega_{\langle\langle \bullet, \bullet\rangle\rangle} \quad \alpha, \beta \in \Omega^k(M_d), \tag{2.4}$$

[3]This means a manifold M is endowed by non-degenerate symmetric bilinear rank 2-tensor field $\langle\langle \bullet, \bullet \rangle\rangle \in \Gamma(SM)$ such that $\langle\langle \bullet, \bullet \rangle\rangle : \chi(M) \times \chi(M) \to \mathbb{R}$.

where $(\bullet, \bullet) : \Omega^k(M_d) \otimes \Omega^k(M_d) \to \Omega^0(M_d)$ is a non-degenerate symmetric bilinear form induced by $\langle\langle \bullet, \bullet \rangle\rangle$ and $\wedge : \omega^m(M_d) \otimes \omega^n(M_d) \to \omega^{m+n}(M_d)$ is the wedge product.

Both for discrete and continuous symmetry groups, we can define $D^g_{d-1}(\mathcal{B}, \Sigma_{(d-1)})$ by cutting space-time along $\Sigma_{(d-1)}$ and inserting a group transformation, in the complete set of states for the Hilbert space, associated to $D^g_{d-1}(\mathcal{B}, \Sigma_{(d-1)})$. The transformations satisfy the group law

$$D^g_{d-1}(\mathcal{B}, \Sigma_{(d-1)}) \circ D^{g'}_{d-1}(\mathcal{B}, \Sigma_{(d-1)}) = D^{g''}_{d-1}(\mathcal{B}, \Sigma_{(d-1)}) \tag{2.5}$$

where $g \cdot g' = g''$ and $\circ$ is the composition of operators. Before discussing how the symmetry transformation is implemented let us recall what a linear representation of a group is. A representation of a group G on a vector space V over a field K is a group homomorphism from $(G, \cdot)$ to $(\mathrm{GL}(V, K), \times)$

$$\rho \colon G \to \mathrm{GL}\,(V, K) \quad \textit{such that} \quad \rho(g \cdot g') = \rho(g) \times \rho(g'), \qquad \textit{for all} \;\; g, g' \in G. \tag{2.6}$$

V is called the representation space and the dimension of V is called the dimension of the representation. The symmetry transformation is implemented when the hypersurface $\Sigma_{(d-1)}$ is deformed and it crosses an operator $\mathcal{O}^{(0)}$ charged under the symmetry group transformation

$$D^g_{d-1}(\mathcal{B}, \Sigma_{(d-1)})\mathcal{O}^{(0)} = \rho(g)\mathcal{O}^{(0)} D^g_{d-1}(\mathcal{B}, \Sigma_{(d-1)}); \tag{2.7}$$

said in other words, the operator $D^g_{d-1}(\mathcal{B}, \Sigma_{(d-1)})$ implements the 0-form symmetry transformation as we cross $\Sigma_{(d-1)}$.

2.2 Higher form symmetry and their generalities

At this point the generalization to higher form symmetries is quite straightforward. A q-form symmetry is given by the existence of a topological operator associated with the riemannian manifold $(\Sigma_{(d-q-1)}, \langle \bullet, \bullet \rangle) \subset (M_d, \langle\langle \bullet, \bullet \rangle\rangle)$, $D^g_{d-q-1}(\mathcal{B}, \Sigma_{(d-q-1)})$ with $g \in G$ where G is global symmetry group and we are considering a d-dimensional theory. The charge objects are not particles but q-dimensional extended object: a sort of branes $\mathcal{O}^{(q)}$. However these branes are very different form branes in string theory since these ones are dynamical.

We stress that the discussion on the homotopical, and not topolocigal, invariance is the same as before and the correlator $\langle ... D^g_{d-q-1}(\mathcal{B}, \Sigma_{(d-q-1)}) ... \rangle$ does not depend from the choice of the representative in $\pi_{d-q-1}(M_d)$. These defects can fuse and the more obvious generalisation of 2.5 is

$$D^g_{d-q-1}(\mathcal{B}, \Sigma_{(d-q-1)}) \circ D^{g'}_{d-q-1}(\mathcal{B}, \Sigma_{(d-q-1)}) = D^{g''}_{d-q-1}(\mathcal{B}, \Sigma_{(d-q-1)}) \tag{2.8}$$

where $g \cdot g' = g''$ and $\circ$ is the composition of operators; however we may also require something different relaxing invertibility as we will see in chapter 4. From 2.8 we can show that the group $(G, \cdot)$ must be abelian, i.e. $g \cdot g' = g' \cdot g$. Indeed, the ordering of the operators in 2.8 can be studied by inserting the two operators at slightly different times, say t_1 and t_2. For 0-form symmetries, the manifold $\Sigma_{(d-q-1)} = \Sigma_{(d-1)}$ is of codimension 1 and the operators $D^g_{d-q-1}(\mathcal{B}, \Sigma_{(d-q-1)})$ at the different times might not commute since we have no dimension to deform the manifold; hence G can be non-Abelian. On the other hand, for $q > 0$ the manifold $\Sigma_{(d-q-1)}$ at time t_1 can be continuously deformed to $\Sigma_{(d-q-1)}$ at time t_2 since we have some transverse dimensions to deform it. Note that in order to follow this idea we need to can deform the manifold in transverse dimensions but this could be not possible even in the case $q > 0$ if the topology of M_d is not trivial.

In the continuous case $D^g_{d-q-1}(\mathcal{B}, \Sigma_{(d-q-1)})$ can be obtained by exponentiating the charge $Q(\Sigma_{(d-q-1)})$ constructed from the Noether $(q+1)$-form closed current $j^{(q+1)}$

$$Q(\Sigma_{(d-q-1)}) = \int_{\Sigma_{(d-q-1)}} \star j^{(q+1)}. \tag{2.9}$$

The analogous of 2.7 for the higher for case is

$$D^g_{d-q-1}(\mathcal{B}, \Sigma_{(d-q-1)}) \mathcal{O}^{(0)} = \rho^{\#(\mathcal{I})}(g) \mathcal{O}^{(q)} D^g_{d-q-1}(\mathcal{B}, \Sigma_{(d-q-1)}); \tag{2.10}$$

where $\mathcal{I}$ is the set of intersections between the q-dimensional extended operator $\mathcal{O}^{(q)}$ and the codimension $(q-1)$ manifold $\Sigma_{(d-q-1)}$.

A tool for studying global symmetry is to couple to background gauge fields; these are $(q+1)$-form gauge fields $A^{(q+1)}$ and lead to a coupling term in the action

$$S[A^{(q+1)}] \propto \int_{M_d} A^{(q+1)} \wedge \star j^{(q+1)}. \tag{2.11}$$

Current conservation means that $S[A^{(q+1)}]$ is invariant under background gauge transformations $A^{(q+1)} \to A^{(q+1)} + dB^{(q)}$, however the invariance can be violated by 't Hooft anomalies.

2.2.1 Spontaneous symmetry braking of 1-form symmetries

Just as an ordinary global symmetry can be spontaneously broken, so can higher-form symmetries and we will focus mostly on 1-form symmetries. We will use the behavior of large Wilson and t' Hooft loops as the diagnostic of such breaking. We interpret an area law for a charged loop operator as reflecting the fact that the corresponding 1-form symmetry is unbroken. Indeed, when a 1-form global symmetry is unbroken, the charged states are strings and they lead to an area law for some loop operators while if the symmetry is spontaneously broken, there are no such strings and hence there is no area law but perimeter one. To be more precisely a 1-form global symmetry G can break to a subgroup K. In that case the loops charged under K exhibit area law while the loops charged under G but uncharged under K exhibit a perimeter law.

Following the standard Goldstone argument [5] with little variations it is possible to show that when a continuous 1-form global symmetry is spontaneously broken the system should have a Goldstone boson and This Goldstone boson is a massless photon. This open an interesting question: the photon is a gauge connection of a Goldstone boson?

3 Higher group symmetries

A right question at this point would be: "is this possible and what happens if higher form symmetries mix?". The question is not so simple and lead to the concept of n-group if to mix are higher form symmetries where the highest form is a $(n-1)$-form. Let us focus on the case of 0-form symmetry and 1-form symmetry mixing for two reasons. On the one hand, the general mathematical definition and understanding of n-group is matter of active mathematical research while, on the other hand, reliable example lie in this restricted class. Therefore, before going on, let us discuss what an 2-group is. By definition a 2-group is a monoidal category $\mathbf{G}$ in which every morphism is invertible and every object has a weak inverse. Let us try to understand what does it mean. First of all a category $\mathbf{C}$ consists of

- a class $\mathrm{ob}(\mathbf{C})$ of objects;
- a class $\mathrm{hom}(\mathbf{C})$ of morphisms between the objects;
- a source object class function $\mathrm{dom}\colon \mathrm{hom}(\mathbf{C}) \to \mathrm{ob}(\mathbf{C})$;
- target object class function $\mathrm{cod}\colon \mathrm{hom}(\mathbf{C}) \to \mathrm{ob}(\mathbf{C})$;
- for every three objects a, b and c, a binary operation $\mathrm{hom}(a,b) \times \mathrm{hom}(b,c) \to \mathrm{hom}(a,c)$ called composition of morphisms, where $\mathrm{hom}(a,b)$ denotes the subclass of morphisms f in $\mathrm{hom}(\mathbf{C})$ such that $\mathrm{dom}(f) = a$ and $\mathrm{cod}(f) = b$. Such morphisms are often written as $f : a \to b$;

such that the following axioms hold:

- if $f : a \to b,\ g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$ (associativity);
- for every object x, there exists a morphism $id_x : x \to x$ called the identity morphism for x, such that every morphism $f : a \to x$ satisfies $id_x \circ f = f$, and every morphism $g : x \to b$ satisfies $g \circ id_x = g$ (identity).

The class of all sets (as objects) together with all functions between them (as morphisms), where the composition of morphisms is the usual function composition, forms a category called **Set**. Other examples are:

1. the category of groups, **Grp**, where the objects are groups, the morphisms are group homomorphisms and the usual function composition as composition;

2. the category of representations, **Rep**(G), where objects are pairs (V, f) of vector spaces V over F and representations f of G on that vector space, the morphism are equivariant maps[4] and the composition is the usual composition of functions.

A this point, a moniodal category $\mathbf{G}$ equipped with a bifunctor[5]

$$\otimes : \mathbf{G} \times \mathbf{G} \to \mathbf{G} \tag{3.1}$$

that is associative up to a natural isomorphism, and an object I that is both a left and right identity for $\otimes$, again up to a natural isomorphism. This means that

$$x \otimes (y \otimes z) \cong_\alpha (x \otimes y) \otimes z, \tag{3.2}$$

and

$$I \otimes x \cong_\lambda x, \quad x \otimes I \cong_\rho x. \tag{3.3}$$

A 2-group is a monoidal category $\mathbf{G}$ such that every morphism is invertible and every object has a weak inverse, i.e. every object x is an object y such that $x \otimes y$ and $y \otimes x$ are both isomorphic to the unit object I.

These kind of structures emerge naturally in the fusion of defects, specifically 2-group structure can emerges in $3d$ when three 2-dimensional defects, associated to a 0-form symmetry, fuse: the three planes intersect each others in one point and this point can be interpreted as the fusion of a 2-dimensional defect with a 1-dimensional one associated with a 1-form symmetry.

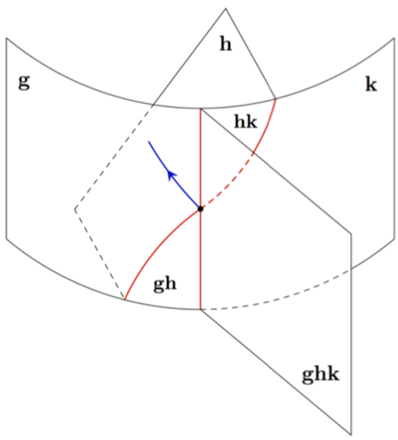

Figure 2: Fusion of bidimensional defects and the emergence of a 1-dimensional one.

[4]A function is said equivariant if its domain and codomain are acted on by the same group G and if the function commutes with the action of the group

[5]A functor is a map between categories and a bifunctor is a functor that has a domain a product category.

3.1 The example of $4d$ spQED with N_f fermions

Adding charged fermions of charge $\tilde{Q}$ produce an 3-form current $J_e^{(3)}$ and the Maxwell equation become

$$\begin{aligned} dF^{(2)} &= 0^{(2)} \\ d\star F^{(2)} &= J_E^{(3)} \end{aligned} . \tag{3.4}$$

Therefore the magnetic Noether current $j_M^{(3)} = \star F^{(2)}$ is no longer conserved and the magnetic $U(1)$ 1-form symmetry is explicitly broken. We have only the electric one. However, spQED has a flavor 0-form symmetry given by $SU(N_f)_R \times SU(N_f)_L$ and looking at the triangle Feynmamn diagram we note a 2-group symmetry structure. Indeed the vanishing of the triangle diagram (flavor)^{2}gauge is a sign that the flavor symmetry is unbroken but there is a deformation of the current algebra between the flavor 0-form symmetry and the 1-form symmetry. Therefore, the mixing of the symmetries is encoded in the correlator between two flavor 1-form currents and the 2-form current $\langle j^{(1)} j^{(1)} j^{(2)} \rangle \propto k$. However, can also be encoded in the properties of background fields. Indeed, introducing the appropriate $B^{(1)}$ and $C^{(2)}$ backgrounds gauge fields we have a sort of Green-Schwarz mechanism such that the right gauge transformations are

$$B^{(1)} \to B^{(1)} + d\lambda^{(0)}, \quad C^{(2)} \to C^{(2)} + d\Lambda^{(1)} + \frac{k}{2\pi} \lambda^{(0)} dB^{(1)}, \tag{3.5}$$

With this modified gauge transformation the generating functional $Z[B^{(1)}, C^{(2)}]$ is now anomaly-free. The pair $(B^{(1)}, C^{(2)})$ together with the gluing rule specified via the gauge transformations above form a so-called 2-connection on a 2-group bundle.

4 Non-invertible symmetries

In the previous sections we saw topological defect operators satisfying a group law fusion. however, in general, not all topological defect operators satisfy a fusion rule of that kind, we define non-invertible codimension $q+1$ defect operators, one satisfying the following fusion rule

$$D^a_{d-q-1}(\mathcal{B}, \Sigma_{(d-q-1)}) \circ D^b_{(d-q-1)}(\mathcal{B}, \Sigma_{(d-q-1)}) = \sum_c N^c_{ab} D^c_{d-q-1}(\mathcal{B}, \Sigma_{(d-q-1)}), \tag{4.1}$$

therefore there may not exist any topological defect operator $D^b_{d-q-1}(\mathcal{B}, \Sigma_{(d-q-1)})$ such that $D^a_{d-q-1}(\mathcal{B}, \Sigma_{(d-q-1)}) \circ D^b_{d-q-1}(\mathcal{B}, \Sigma_{(d-q-1)})$ is the identity defect operator. In previous cases, the existence of such a defect operator was guaranteed by the group law structure of the fusion; indeed whereas multiplication of two elements of a group always produces a unique third element, here we produce a superposition of elements, weighted by fusion multiplicities N^c_{ab}. Such symmetries are called categorical symmetries, non-invertible symmetries or fusion category symmetries. The term higher categorical refers to the mathematical structure of this symmetries, that is not those of a group or an n-group but those of an higher category. Essentially an n-category $\mathbf{C}^{(n)}$ has n-levels: at the first level, we have objects of the category, which are also called 0-morphisms; at the second level, we have 1-morphisms between objects; at the third level, we have 2-morphisms between 1-morphisms; continuing in this fashion, at the i-th level for $2 \leq i \leq n$, we have $(i-1)$-morphisms between $(i-2)$-morphisms. In the case of a d-dimensional QFT, called Ξ, we can built up a $(d-1)$-category $\mathcal{C}_\Xi$ which we refer to as the symmetry category of Ξ since it taken into account all possible higher form symmetries.

In this $\mathcal{C}_\Xi$ construction the objects correspond to topological codimension-1 defect operators of Ξ. Since the sum of different topological codimension-1 defect operators is again a topological codimension-1 defect operator we have an additive structure

$$\oplus_i n_i D^i_{d-1}(\mathcal{B}, \Sigma^i_{(d-1)}) = D_{d-1}(\mathcal{B}, \Sigma_{(d-1)}); \tag{4.2}$$

moreover, due to the fusion rule, we have also a multiplicative structure. The 1-morphisms of $\mathcal{C}_\Xi$ correspond to topological codimension-2 defect operators living at the intersection of two topological codimension-1 defect operators. They have an additive and a multiplicative structure fro the same reasons as objects have them. Continuing inductively, we define p-morphisms of $\mathcal{C}_\Xi$ correspond to topological codimension-p defect operators living at the intersection of two topological codimension-$(p-1)$ defect operators and they have the same structure as before. We stress that this chain of multiplicative structure is equivalent to say that the symmetry category has monoidal structure.

4.1 The generalization of charges: the p-charges

We introduced category and higher categories, at this point we have the mathematical tools to introduce the concept of p-charges. They are generalizations of the standard charges that we are used to compute in presence of a symmetry [6]. The key to unlocking the full utility of generalized, in particular non-invertible, symmetries is to understand their action on local and extended operators of various dimensions. Said anothe way, the key is to determine the generalized charges carried by operators in a QFT with generalized global symmetries. The role that representation theory plays for groups, is replaced in this context by higher-representations, which are intimately related to the categorical nature of the symmetries. At the moment only invertible symmetries find a place in this room and the case of non-invertible will be addressed in future.

We call p-charges the generalized charges of p-dimensional operators. The authors classify the possoble p-charges of various higher form symmetry and higher groups in such a way to make a bridge with the fundamental idea that charges are representation of the symmetry group. Hovewer, the standard paradigm of "p-dimensional operators are charged under p-form symmetries" turns out to only the tip of the iceberg. Authors find that a p-form symmetry generically acts on defects operators of $(q \geq p)$-dimensions. To describe this kind of charges standard representations are not enough and we need to introduce some more mathematical tools: higher representations.

In full generality authors propose the following paradigm that we will discuss in a while:

1. p-charges of a $G^{(p)}$ p-form symmetry are representations of the symmetry group $G^{(p)}$;

2. p-charges of a $G^{(0)}$ 0-form symmetry are $(p+1)$-representations of the group $G^{(0)}$;

3. p-charges of a $G^{(q)}$ q-form symmetry are $(p+1)$-representations of the associated $(q+1)$-group $\mathbf{G}^{(q+1)}_{G^{(q)}}$;

4. p-charges of a $\mathbf{G}^{(q)}$ q-group symmetry are $(p+1)$-representations of the q-group $\mathbf{G}^{(q)}$.

These four cases cover all possibility for invertible symmetries at least. Let us explain better case by case:

1. this is the standard case, therefore when i have a p-form symmetry the charged defect operators are p-dimensional. Note however that a single group $G^{(0)}$ can be recast in a category $\mathbf{G}_{G^{(0)}}$ such that there is a single object. As there is a single object, all morphisms are in fact endomorphisms of this object labelled by a $g \in G^{(0)}$ and the composition of endomorphisms follows the group multiplication law.

2. we need to introduce the tool of the higher representation. Before do this, let us rephrase the concept of standard representations in the real of categories. Finite dimensional vector spaces form a linear category called **Vec** while whose objects are vector spaces and morphisms are linear maps between vector spaces. In this context, a representation ρ of $G^{(0)}$ can be viewed as a functor

$$\rho^{(1)} : \mathbf{G}_{G^{(0)}} \to \mathbf{Vec}, \tag{4.3}$$

therefore $\rho^{(1)}$ maps the single object of $\mathbf{G}_{G^{(0)}}$ to an object V of **Vec**, which is the underlying vector space for the representation ρ and maps the endomorphisms of the single object of $\mathbf{G}_{G^{(0)}}$ to endomorphisms of V. Now a $(p+1)$-representation is a functor

$$\rho^{(p+1)} : \mathbf{G}^{(p+1)}_{G^{(0)}} \to \mathbf{Vec}^{(p+1)}. \tag{4.4}$$

Therefore the action of a 0-form symmetry on a $(p > 0)$-dimensional defect operators is classified by the higher-representations.

3. Before we defined a 2-group but here we need higher group. A $(q+1)$-group $\mathbf{G}^{(q+1)}$ is a structure describing r-form symmetry groups for $0 \leq r \leq p$ along with possible interactions, i.e mixings, between the different r-form symmetry groups. Now, a q-form symmetry group $G^{(q)}$ naturally forms a $(q+1)$-group $\mathbf{G}^{(q+1)}_{G^{(q)}}$ whose r-form symmetry groups are all trivial except for $r = q$;

4. this seems standard but higher representations of higher groups are not well understood.

4.2 Application to quantum gravity through Swampland conjectures

Although the landscape of quantum gravity theories may be vast, certain features seem to be universally true of all such theories. These features are, in some sense, formalized in the context of the swampland program where we search for those effective low-energy physical theories which are not compatible with quantum gravity. The Swampland program aims to delineate the theories of quantum gravity by identifying the universal principles shared among all theories compatible with gravitational UV completion. The program was initiated arguing that string theory suggests that the Swampland is in fact much larger than the string theory landscape [7]. One such feature is the absence of global symmetries, including q-form global symmetries, for which the charged operators are supported on manifolds of dimension q. Another such feature is completeness of the spectrum, i.e. the presence of particles (or multiparticle states) transforming in every representation of the gauge group. It is possible to show that in a gauge theory with connected and compact gauge group G, which has a 1-form electric symmetry associated to $Z(G)$ that it is explicitly broken to a subgroup in the presence of charged matter, that it is broken completely if and only if the spectrum is complete. Thus, in such a theory, absence of the 1-form electric symmetry is in bijective correspondence with completeness of the spectrum. However, this correspondence between the absence of global symmetries and completeness does not hold in general, indeed a finite, non-abelian gauge group G, such as S_4, may have a trivial center, so it does not have a 1-form electric symmetry even if its spectrum is incomplete [8]. Nevertheless, it is possible to generalize this no global symmetries–completeness correspondence in the real of non.invertible symmetries: Consider a gauge theory with compact gauge group G coupled to a set of matter fields transforming in representations of G. Then the theory is electrically complete (i.e., states exist transforming in all possible representations of G) if and only if there are no codimension 2-topological Gukov-Witten operators in the theory, included the non-invertible ones. Therefore the completeness of the spectrum is in bijective correspondence with the absence of non-invertible 1-form electric symmetries, which are characterized by the presence of topological, non-invertible codimension-2 operators.

4.3 Build up non-invertible symmetries from invertible ones

In the following we show two methods which allow to build examples of non-invertible symmetries starting from theories with generalized symmetries: the gauiging of a 0-form symmetry of a Topological Quantum Field Theories (TQFT) and the gauging of an outer automorphism 0-form symmetry.

4.3.1 Gauging a 0-form symmetry in a TQFT

A TQFT is a quantum field theory which computes topological invariants; meaning that the observables are homotopical or topological invariants. There existi two type of TQFTs: the Schwarz-type and the Witten type. In the first case the action does not depend on the space-time metric while in the second case it depends but a topological twist[6] it turns out to be metric independent.

The basic idea [9] is to have a d-dimensional theory $\mathfrak{T}$ with a global 0-form symmetry $\Gamma^{(0)}$ and we stack this theory with a $(p < d)$-dimensional topological TQFT T, which also has a global $\Gamma^{(0)}$ 0-form symmetry, and gauge $\Gamma^{(0)}$. This gauging has the effect of coupling the d-dimensional and the p-dimensional systems together, such that T becomes a topological defect operator $D_p(\mathcal{B}, \Sigma_{(p)})$ i.e. a symmetry generator in the gauged theory $\mathfrak{T}/\Gamma^{(0)}$ that can be non-invertible. These kind of defect operators are called Θ-defects.
The path to follow is to determine what are the possible TQFTs with $G^{(0)}$ 0-form symmetry, understanding the fusion rules of these TQFT and gauging the $G^{(0)}$ 0-form symmetry to get topological Θ-defect operators with the fusion rule determined before. The gauging procedure is represented in the following figure.

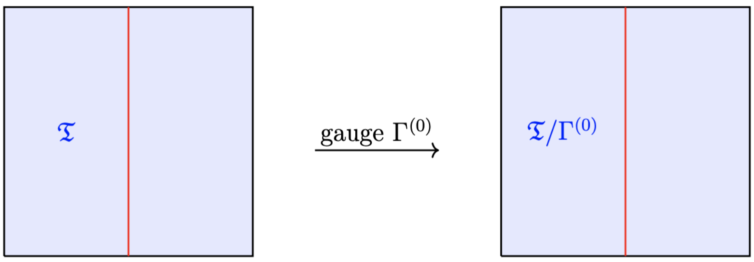

Figure 3: The gauging of a 0-form symmetry in a theory $\mathfrak{T} \times T$.

Let us focus on a 3d $\mathfrak{T}$ QFT and 2d TQFT. Given a 0-form symmetry with group $G^{(0)}$, the possible TQFTs are those that:

1. preserve completely the group $G^{(0)}$;

[6]Topological twist is a procedure for producing lagrangians for topological quantum field theories from non-topological but supersymmetric QFTs. This is achieved requiring that the Lorentz symmetry generators that appear in the supersymmetry algebra simultaneously rotate the physical spacetime and also act on one of the R-symmetries.

2. the group $G^{(0)}$ is spontaneously broken to a subgroup.

We label a TQFT as $T^{(H)}$ where H is the unbroken subgroup of the group $G^{(0)}$; the fusion rule is

$$T^{(H)} \otimes T^{(K)} = n_{H,K} T^{H \cap K} \tag{4.5}$$

with

$$n_{H,K} = \frac{|\frac{G^{(0)}}{H}||\frac{G(0)}{K}|}{|\frac{G(0)}{H \cap K}|}. \tag{4.6}$$

When we gauge the $G^{(0)}$ 0-form symmetry we get a set of Θ-defect $D_p^{(\frac{G(0)}{H})}(\mathcal{B}, \Sigma_{(p)})$ that satisfy fusion rules 4.5 and if $n_{H,K} \neq 1$ these are non-invertible Θ-defect underlying a non-invertible symmetry.

4.3.2 Gauging an outer automorphism 0-form symmetry

The outer automorphism group of a group G is the quotient, $\frac{\mathrm{Aut}(G)}{\mathrm{Inn(G)}}$, where $\mathrm{Aut}(G)$ is the automorphism group of G and Inn(G) is the subgroup consisting of inner automorphisms, i.e. automorphism arising from conjugation $\phi_g \colon G \to G$ where $\phi_g(x) := g^{-1}xg,\ g \in G$. In the case of semisimple Lie algebras the automorphism group equals the group of diagram automorphisms.

Given a 1-form symmetry and an outer automorphism 0-form symmetry we can recast the information in the real of category and gauging the 0-form symmetry we can get a category describing non-invertible symmetries [10].

5 Conclusion

A Wilson and t'Hooft lines

Let us discuss Wilson and t' Hooft lines in $4d$ gauge theories. Wilson loops are gauge invariant operators arising from the parallel transport of gauge variables around closed loops. They encode all gauge information of the theory, allowing for the construction of loop representations which fully describe gauge theories in terms of these loops. The definition of a Wilson line is

$$W[x_i, x_f] = \mathcal{P} \exp\left(i \int_{x_i}^{x_f} A_\mu dx^\mu\right), \tag{A.1}$$

where $\mathcal{P}$ is the path-ordering operator, which is unnecessary for abelian theories. The trace of closed Wilson lines is a gauge invariant quantity known as the Wilson loop

$$W[\gamma] = Tr\left[\mathcal{P} \exp\left(i \oint_\gamma A_\mu dx^\mu\right)\right] = Tr\left[\mathcal{P} \exp\left(i \oint_\gamma A^{(1)}\right)\right]. \tag{A.2}$$

We note that space-time loops are related to closed loop of electric flux while spatial loop measures the magnetic flux through the loop. Indeed, using Stokes theorem, we have

$$i \oint_\gamma A_\mu dx^\mu = i \oint_\gamma A^{(1)} = \int_\Sigma F^{(2)} = i \int_\Sigma F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{A.3}$$

with $\partial\Sigma = \gamma$ and since

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{A.4}$$

if the loop lie on a space-time surface we have the electric flux while if lie on a space surface we have the magnetic flux.

A very closely related concept is that of 't Hooft loop; is is a magnetic analogue of the Wilson loop for which space-time loops give rise to thin loops of magnetic flux. Indeed a 't Hooft loop is defined as

$$T[\gamma] = Tr\left[\mathcal{P}\exp\left(i \oint_\gamma \tilde{A}_\mu dx^\mu\right)\right] = Tr\left[\mathcal{P}\exp\left(i \oint_\gamma \tilde{A}^{(1)}\right)\right], \tag{A.5}$$

where $\tilde{A}^{(1)}$ is the magnetic photon, i.e the dual connection. Since

$$\star F^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \tag{A.6}$$

and

$$i \oint_\gamma \tilde{A}_\mu dx^\mu = i \oint_\gamma \tilde{A}^{(1)} = \int_\Sigma \star F^{(2)} = i \int_\Sigma \star F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{A.7}$$

we note that if the loop lie on a space-time surface we have the magnetic flux while if lie on a space surface we have the electric flux.

The two objects are useful since they are order parameters for the gauge theory.The can have area law where the expectation value goes as

$$\sim e^{-aA[\gamma]}, \tag{A.8}$$

with $A[\gamma]$ being the area enclosed by the loop or perimeter law where the expectation value goes as

$$\sim e^{-aL[\gamma]}, \tag{A.9}$$

with $L[\gamma]$ being the perimeter of the loop.

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