# Exercises for mathematical methods of physics

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# 1 Complex and analytic functions

## 1.1 de Moivre formula and integer powers

Given the complex number $z = 1 - \sqrt{3}i$ compute the number $z^6$ and write the result in cartesian coordinates.

Given the polar form of a complex number $z = \rho e^{i\theta}$ with $\rho$ its modulus and $\theta$ its argument,

$$\rho = \sqrt{Re(z)^2 + Im(z)^2},\ Arg(z) = \begin{cases} arctang\big(\frac{Im(z)}{Re(z)}\big) & if\ Re(z) > 0 \\ arctang\big(\frac{Im(z)}{Re(z)}\big) + \pi & if\ Re(z) < 0 \cup Im(z) \geq 0 \\ arctang\big(\frac{Im(z)}{Re(z)}\big) - \pi & if\ Re(z) < 0 \cup Im(z) < 0 \end{cases}, \tag{1.1}$$

we can compute

$$z^n = \rho^n e^{in\theta}; \tag{1.2}$$

where now the modulus is $\rho^n$ while the argument is $n\theta$. Using Euler formula we can write

$$z^n = \rho^n[cos(n\theta) + isin(n\theta)], \tag{1.3}$$

this is the de Moivre formula. Let us use it to compute $z^6$ with $z = 1 - \sqrt{3}i$. The modulus is $\rho = \sqrt{1+3} = \sqrt{4} = 2$ while its argument is $Arg(z) = arctang(-\frac{\sqrt{3}}{1}) = -\frac{\pi}{3}$. Therefore the we can compute the power

$$z^6 = 2^6\left[cos\left(-\frac{6\pi}{3}\right) + isin\left(-\frac{6\pi}{3}\right)\right] = 64. \tag{1.4}$$

## 1.2 Complex logarithm and exponential

Compute the natural logarithm and the exponential of the number $z = -2\sqrt{3} + 2i$.

The complex logarithm is given by

$$log(z) = log(\rho) + i(Arg(z) + 2k\pi) \quad k \in \mathbb{Z} \tag{1.5}$$

where the principal branch is $-\pi < Arg(z) < \pi$ and without the term $2ik\pi$, while the complex exponential is given by

$$e^z = e^{Re(z)}e^{iIm(z)}. \tag{1.6}$$

In our case we have $Re(z) = -2\sqrt{3}, Im(z) = 2$ and

$$\rho = \sqrt{12+4} = \sqrt{14}, \quad Arg(z) = arctang\left(-\frac{2}{2\sqrt{3}}\right) + \pi = -\frac{\pi}{6} + \pi = \frac{5}{6}\pi; \tag{1.7}$$

therefore

$$log(z) = log(\sqrt{14}) + i\frac{5}{6}\pi = \frac{1}{2}log(14) + i\frac{5}{6}\pi, \quad e^z = e^{-2\sqrt{3}}e^{2i}. \tag{1.8}$$

All the other branches of the logarithm are given by

$$log(z) = log(\sqrt{14}) + i\frac{5}{6}\pi = \frac{1}{2}log(14) + i\frac{5}{6}\pi + 2ik\pi, \quad k \in \mathbb{Z}. \tag{1.9}$$

## 1.3 Complex powers and root

Compute the powers $z^e$, $z^i$ and the root $\sqrt[4]{z}$, of $z = 2 + 2i$.

Let us compute modulus and argument of $z$: $\rho = \sqrt{4+4} = \sqrt{8}$ and $Agr(z) = arctang(1) = \frac{\pi}{4}$. The power for a complex exponent is given by the polyhydrome function

$$(z^\alpha)_k = e^{log(z^\alpha)} = e^{\alpha log(z)} = e^{\alpha log(\rho)} e^{i\alpha(Arg(z)+2k\pi)} \quad with\ k \in \mathbb{Z}; \tag{1.10}$$

therefore

$$\begin{aligned}(z^e)_k &= e^{elog(\sqrt{8})} e^{ie\left(\frac{\pi}{4}+2k\pi\right)} = 8^{\frac{e}{2}}\left[cos\left(e\left(\frac{\pi}{4}+2k\pi\right)\right) + isin\left(e\left(\frac{\pi}{4}+2k\pi\right)\right)\right];\\ (z^i)_k &= e^{ilog(\sqrt{8})} e^{i^2\left(\frac{\pi}{4}+2k\pi\right)} = e^{-\left(\frac{\pi}{4}+2k\pi\right)}[cos(log(\sqrt{8})) + isin(log(\sqrt{8}))].\end{aligned} \tag{1.11}$$

The root is also a polyhydrome function and the values of the root constitute the vertices of a regular polygon inscribed in a circumference of radius $\sqrt[n]{\rho}$ in the complex plane; it is given by

$$w_k = \rho^{\frac{1}{n}} e^{i\frac{Arg(z)+2k\pi}{n}} \quad with\ n \in \mathbb{Z}\ and\ k = 0, ..., n-1; \tag{1.12}$$

therefore

$$\begin{aligned} w_0 &= 8^{\frac{1}{8}} e^{i\frac{\frac{\pi}{4}}{4}} = 8^{\frac{1}{8}}\left[cos\left(\frac{\pi}{16}\right) + isin\left(\frac{\pi}{16}\right)\right];\\ w_1 &= 8^{\frac{1}{8}} e^{i\frac{\frac{\pi}{4}+2\pi}{4}} = 8^{\frac{1}{8}}\left[cos\left(\frac{9\pi}{16}\right) + isin\left(\frac{9\pi}{16}\right)\right];\\ w_2 &= 8^{\frac{1}{8}} e^{i\frac{\frac{\pi}{4}+4\pi}{4}} = 8^{\frac{1}{8}}\left[cos\left(\frac{17\pi}{16}\right) + isin\left(\frac{17\pi}{16}\right)\right];\\ w_3 &= 8^{\frac{1}{8}} e^{i\frac{\frac{\pi}{4}+6\pi}{4}} = 8^{\frac{1}{8}}\left[cos\left(\frac{31\pi}{16}\right) + isin\left(\frac{31\pi}{16}\right)\right]; \end{aligned} \tag{1.13}$$

Values of the $n$th root are particularly important for the finite $n$th cyclic group; indeed normalizing them to their common modulus we get the elements of the finite $n$th cyclic group. Moreover the regular polygon these values drown in the complex plane is called cycle graphs in these context and turn out to be very useful in order to understand isomorphisms between these groups. Cyclic groups plays and important role in physics for example in some geometrical constructions in string theory or in the standard models of particle physics.

## 1.4 Analyticity domain of functions

Given the following functions says which is their analyticity domain:

- $f(z) = log((2-z)^2)$;

- $g(z) = z^z$;
- $h(z) = log(cos(z))$;
- $q(z) = \sqrt{z^2 - 1}$;
- $p(z) = \frac{sinh(sin(z))}{z^2+9}$.

Let us assume, in general, the principal branch for the logarithm; therefore the branch cut is along the negative real axis. The logarithm is analytic in $\mathbb{C} \setminus \{z|Re(z) \leq 0 \cup Im(z) = 0\}$ so to find the analyticity domain of $f(z)$ we need to solve the equation

$$(2 - z)^2 = -t \Rightarrow z = 2 \pm i\sqrt{t} \quad t \geq 0. \tag{1.14}$$

Therefore $f(z)$ is analytic in $\mathbb{C} \setminus \{z|z = 2 \pm i\sqrt{t}, t \geq 0\}$.
Function $z^z$ can be rewritten as

$$z^z = e^{zlog(z)}, \tag{1.15}$$

since the exponential is analytic in $\mathbb{C}$, analyticity domain of the function $g(z)$ is the same of $log(z)$, namely $\mathbb{C} \setminus \{z|Re(z) \leq 0 \cup Im(z) = 0\}$.
Thinking as before, for $h(z)$ we need to solve the equation

$$cos(z) = -t \Rightarrow \begin{cases} cos(x)cosh(y) & = -t \\ -sin(x)sinh(y) & = 0 \end{cases} \quad t \geq 0, \tag{1.16}$$

where we used $z = x + iy$, $cos(iy) = cosh(x)$, $sin(iy) = isinh(y)$ and

$$cos(x+iy) = cos(x)cos(iy) - sin(x)sin(iy) = cos(x)cosh(y) - isin(x)sinh(y). \tag{1.17}$$

Form the second equation we have

$$x = k\pi \ k \in \mathbb{Z} \ \ or \ \ y = 0; \tag{1.18}$$

since $cosh(y) \geq 1$ we need to discard solutions with $k = 2n$ with $n \in \mathbb{Z}$ otherwise the first equation has no solution (since $cos(x)$ would be positive). On the other hand solutions with $k = 2n + 1$ with $n \in \mathbb{Z}$ are ok and ($cos(x)$ would be negative)

$$cosh(y) = t \ \ with \ x = (2n + 1)\pi \ and \ n \in \mathbb{Z}, \tag{1.19}$$

whose solution is $y \in (-\infty, +\infty)$; these are numbers of the form

$$I := \{z|z = (2n + 1)\pi + iy \ \ with \ n \in \mathbb{Z} \ and \ y \in (-\infty, +\infty)\}. \tag{1.20}$$

If $y = 0$ then $cosh(y) = 1$ and

$$cos(x) = -t \Rightarrow x = arccos(-t) \tag{1.21}$$

and since $t \geq 0$, $x \in [\frac{\pi}{2}, \pi]$ (because the argument must be smaller than 1 and so $-t \in [-1, 0]$), these are numbers of the form

$$II := \left\{ z | z = x \;\; with\; x \in \left[ \frac{\pi}{2}, \pi \right] \right\}. \tag{1.22}$$

in the end, function $h(z)$ is analytic in $\mathbb{C} \setminus \{I \cup II\}$.
Function $q(z)$ can be rewritten as

$$\sqrt{z^2 - 1} = e^{\frac{1}{2} log(z^2-1)}, \tag{1.23}$$

so, again, we have

$$z^2 - 1 = -t \;\; with\; t \geq 0. \tag{1.24}$$

This equation has solutions

$$z^2 = -t + 1 \Rightarrow z = \pm\sqrt{-t+1} \tag{1.25}$$

but we have to pay attention at the case $t \in (1, +\infty)$, so

$$\begin{cases} z = \pm\sqrt{1-t} & if\; t \in [0, 1]; \\ z = \pm i\sqrt{t-1} & if\; t \in (1, +\infty); \end{cases} \tag{1.26}$$

the first case is the real compact $[-1, 1] := \{z = x | x \in [-1, 1]\}$ while the second case is the imaginary axis $Im(z) \setminus \{0\}$. Therefore, our function in analytical in $\mathbb{C} \setminus \{[-1, 1] \cup Im(z) \setminus \{0\}\}$
The last case is simpler. $sin(z)$ and $sinh(z)$ are entire and entire function form a closed algebra under composition; therefore the only problems are the zeros of the denominator

$$z^2 + 9 = 0 \Rightarrow z = \pm 3i. \tag{1.27}$$

Therefore, $p(z)$ is analytical in $\mathbb{C} \setminus \{-3i, 3i\}$.

## 1.5 Analytic functions

Given the functions $f(z) = z + e^z$, $f(\bar{z})$ and $g(z) = \frac{\bar{z}}{z}$ say if they are analytic functions or not.

Let us start writing the real and imaginary parts of our functions

$$\begin{aligned} f(z) &= x + iy + e^{x+iy} = x + iy + e^x[cos(y) + isin(y)] = x + e^x cos(y) + i[y + e^x sin(y)]; \\ g(z) &= \frac{x - iy}{x + iy}\frac{x - iy}{x - iy} = \frac{x^2 - y^2 - 2ixy}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2} - i\frac{2xy}{x^2 + y^2}. \end{aligned} \tag{1.28}$$

Consider $f(z)$, its real and imaginary parts are

$$u = x + e^x cos(y), \quad v = y + e^x sin(y), \tag{1.29}$$

these are differentiable function in $\mathbb{R}^2$ and we only need to check Cauchy-Riemann conditions. These are obviously satisfied since $\frac{\partial f(z)}{\partial \bar{z}} = 0$, but let us do the full computation:

$$\frac{\partial v}{\partial x} = e^x sin(y), \quad \frac{\partial u}{\partial y} = -e^x sin(y) \Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{1.30}$$

and

$$\frac{\partial v}{\partial y} = 1 + e^x cos(y), \quad \frac{\partial u}{\partial x} = 1 + e^x cos(y) \Rightarrow \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}. \tag{1.31}$$

In the and $f(z)$ is an analytic function. Let us see $g(z)$, its real and imaginary parts are

$$u = \frac{x^2 - y^2}{x^2 + y^2}, \quad v = -\frac{2xy}{x^2 + y^2}, \tag{1.32}$$

these function are differentiable everywhere in $\frac{\mathbb{R}^2}{\{0,0\}}$. However we expect that $g(z)$ is not analytic since it depends on $\bar{z}$, let us check explicitly

$$\frac{\partial v}{\partial x} = \frac{2x^2y - 2y^3}{(x^2+y^2)^2}, \quad \frac{\partial u}{\partial y} = \frac{2x^2y + 2y^3}{(x^2+y^2)^2} \Rightarrow \frac{\partial v}{\partial x} \neq -\frac{\partial u}{\partial y}. \tag{1.33}$$

Last but not least, $f(\bar{z})$; obviously this is not an analytic function since $f(z)$ is. Anyway $f(\bar{z})$ is antianalytic (this is a universal property: if $f(z)$ is analytic then $f(\bar{z})$ is antianalytic) and satisfies a set of revisited Cauchy-Riemann conditions (where the minus sing is in the other equation).

## 1.6 Construction of analytic or antianalytic functions

Consider the real functions $f(x,y) = x^2 + xe^y$, $g(x,y) = cos(x) + y$, $q(x,y) = e^{3x} sin(3y)$ and $h(x,y) = x^2 - y^2 + x$. which of these functions can be the real or imaginary part of an analytic or antianalytic function? For those that are, construct the associated analytic or antianalytic complex functions.

We need to check thet they are harmonic function on the real plane, namely they satisfy the Poisson equation $\Delta U(x,y) = 0$. Let us check:

$$\begin{aligned}
\Delta f(x,y) &= (\partial_x^2 + \partial_y^2) f(x,y) = 2 + xe^y \neq 0; \\
\Delta g(x,y) &= (\partial_x^2 + \partial_y^2) g(x,y) = -cos(x) \neq 0; \\
\Delta q(x,y) &= (\partial_x^2 + \partial_y^2) h(x,y) = 9e^{3x} sin(3y) - 9e^{3x} sin(3x) = 0. \\
\Delta h(x,y) &= (\partial_x^2 + \partial_y^2) h(x,y) = 2 - 2 = 0.
\end{aligned} \tag{1.34}$$

Only $q(x,y)$ and $h(x,y)$ are harmonic functions, let us construct the analytic and antianalytic complex functions associated. Let us assume that $q(x,y)$ is the real part of an analytic complex function, $q(x,y) \equiv u(x,y) = e^{3x} sin(3y)$; essentially we have to solve the Cauchy-Riemann conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{1.35}$$

We have

$$\frac{\partial v}{\partial y} = 3e^{3x} sin(3y) \tag{1.36}$$

and

$$\frac{\partial v}{\partial x} = -3e^{3x} cos(3y). \tag{1.37}$$

Integrating 1.36 in $y$ we get $v(x,y)$ up to a $x$-dependent arbitrary function

$$v(x,y) = \int 3e^{3x} sin(3y) dy \underbrace{=}_{3y=l} e^{3x} \int sin(l) dl = -e^{3x} cos(3y) + C(x); \tag{1.38}$$

to fix $C(x)$ we impose 1.37, therefore

$$\frac{\partial v}{\partial x} = -3e^{3x} cos(3y) + C'(x) \underbrace{=}_{1.37} -3e^{3x} cos(3y) \Rightarrow C'(x) = 0 \Rightarrow C(x) = C. \tag{1.39}$$

In the and our analytic complex function is

$$\begin{aligned} & f(z) = u(x,y) + iv(x,y) = \\ & = e^{3x} sin(3y) - ie^{3x} cos(3y) + iC = e^{3x}(sin(3y) - icos(3y)) + iC = \\ & - ie^{3x} e^{3iy} + iC = -ie^{3(x+iy)} + iC = -ie^{3z} + iC. \end{aligned} \tag{1.40}$$

Let us repeat the exercise for $h(x,y)$ but using Cauchy-Riemann conditions for antianalytuc functions

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}. \tag{1.41}$$

Assume that $h(x,y)$ is the imaginary part of an antianalytuc function, $h(x,y) \equiv v(x,y) = x^2 - y^2 + x$. We than have

$$\frac{\partial u}{\partial x} = 2y \tag{1.42}$$

and

$$\frac{\partial u}{\partial y} = 2x + 1. \tag{1.43}$$

Integrating 1.42 in $x$ we get $u(x,y)$ up to a $y$-dependent arbitrary function

$$u(x,y) = \int 2y dx = 2xy + C(y); \tag{1.44}$$

to fix $C(y)$ we impose 1.43, therefore

$$\frac{\partial u}{\partial y} = 2x + C'(y) \underbrace{=}_{1.43} 2x + 1 \Rightarrow C'(y) = 1 \Rightarrow C(y) = y + C. \tag{1.45}$$

In the and our antianalytic complex function is

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) = 2xy + y + C + i(x^2 - y^2 + x) = \\ &= 2xy + i(x^2 - y^2) + y + ix + C = i\bar{z}^2 + i\bar{z} + C \end{aligned} \tag{1.46}$$

## 1.7 Remuvable, polar and essential singularities

Given the complex functions:

1. $f(z) = \frac{sin(z-2)}{z-2}$;
2. $g(z) = \frac{e^{\frac{1}{z}}}{z^2}$;
3. $h(z) = \frac{1}{(z-2)^2 z}$;
4. $q(z) = \frac{3z^3+2z^2+z}{3z+2z^2+z^3}$;
5. $\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}} \qquad \gamma \approx 0.577216$;

say which kind of singularity they have and if are meromorphic, entire or none of these.

Let us begin with some reminders. A isolated singularity in $z_0$ is called remuvable if exists finite the limit of the function in this point

$$\lim_{z \to z_0} f(z) = c \in \mathbb{C}; \tag{1.47}$$

therefore, defining a cases function that take the value $c$ at $z = z_0$ we get an analytic function. A singularity in $z_0$ is called a pole of order $n$ if exists finite, in this point, the limit of the function times the monomial $m(z; z_0; n) = (z - z_0)^n$

$$\lim_{z \to z_0} m(z; z_0; n) f(z) = \lim_{z \to z_0} (z - z_0)^n f(z) = c \in \mathbb{C}. \tag{1.48}$$

A singularity in $z_0$ is called essential if does not exist the limit of the function at this point. We remind also the definition of entire and meromorphic function: a function is called entire if and only if it has no singularity in $\mathbb{C}$ but only in $\hat{\mathbb{C}}$ while it is called meromorphic if has only polar singularity in $\mathbb{C}$.
Let us now begin with the first function; this is analytic in $\mathbb{C} \setminus \{2\}$ so we take the limit for $z \to 2$. Since $sin(z-2)$ is analytic we can expand it around $z_0 = 2$ in series and this series must converge; this series is the same of the real function $sin(x)$ and therefore we have (we can use also de l'Hôpital rule, but pay attention: this rule is

due to Lagrange theorem and in general it is not true for a complex function, however if the function is at least meromorphic we can apply safely the rule)

$$\lim_{z\to 2} f(z) = \lim_{z\to 2} \frac{\sum_{k=0}^{\infty}(-1)^k(z-2)^{2k+1}}{(z-2)(2k+1)!} = \lim_{z\to 2}\sum_{k=0}^{\infty}\frac{(-1)^k(z-2)^{2k}}{(2k+1)!} = 1. \tag{1.49}$$

So, we can define an extension of the function $f(z)$ that is analytical all over $\mathbb{C}$:

$$\tilde{f}(z) = \begin{cases} f(z) & if\ z \in \mathbb{C}/\{2\} \\ 1 & if\ z = 2 \end{cases}. \tag{1.50}$$

The second function is $g(z)$, we have a singularity in $z_0 = 0$ and

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} e^{\frac{1}{z}} \lim_{z\to 0}\frac{1}{z^2}; \tag{1.51}$$

both limits diverge but the second one can be "adjusted", indeed

$$\lim_{z\to 0} m(z,0,2)\frac{1}{z^2} = \lim_{z\to 0} z^2\frac{1}{z^2} = 1. \tag{1.52}$$

The real problem is the first limit, indeed there is no hope the limit converge when we multiply it by $m(z,0,n)$ for none $n$; therefore this limit seems to give a essential singularity, let us check it. Write $z = \rho e^{i\theta}$, so

$$\lim_{z\to 0} e^{\frac{1}{z}} = \lim_{(\rho,\theta)\to(0,\theta^*)} e^{\frac{1}{\rho e^{i\theta}}} \tag{1.53}$$

and this limit gives different values for different values of $\theta^*$. Let us exhibit this behavior: choose $\theta^* \equiv \theta_1 = 0$ and $\theta^* \equiv \theta_2 = \frac{\pi}{2}$, we get

$$\lim_{(\rho,\theta)\to(0,0)} e^{\frac{1}{\rho e^{i\theta}}} = \lim_{(\rho,\theta)\to(0,0)} e^{\frac{1}{\rho}} = \infty \tag{1.54}$$

while

$$\lim_{(\rho,\theta)\to(0,\frac{\pi}{2})} e^{\frac{1}{\rho e^{i\theta}}} = \lim_{(\rho,\theta)\to(0,\frac{\pi}{2})} e^{\frac{1}{\rho e^{i\frac{\pi}{2}}}} = \lim_{(\rho,\theta)\to(0,\frac{\pi}{2})} e^{-\frac{i}{\rho}} = \lim_{(\rho,\theta)\to(0,\frac{\pi}{2})}\left[cos\left(\frac{1}{\rho}\right) - isin\left(\frac{1}{\rho}\right)\right] \tag{1.55}$$

which is complex and does not exist.
It is now time of $h(z)$, this seems to have only polar singularities since we have no numerator to expand in series. Singularity point are in $z_0 = 0$ and $z_1 = 2$ so let us consider the limits

$$\lim_{z\to 0} m(z;0;1)h(z) = \lim_{z\to 0} z\frac{1}{z(z-2)^2} = \frac{1}{4}, \tag{1.56}$$

and

$$\lim_{z\to 2} m(z;2;1)h(z) = \lim_{z\to 2}(z-2)\frac{1}{z(z-2)^2} = \lim_{z\to 2}\frac{1}{z(z-2)} = \infty, \tag{1.57}$$

no good novels; let us try with

$$\lim_{z\to 2} m(z;2;2)h(z) = \lim_{z\to 2}(z-2)^2\frac{1}{z(z-2)^2} = \lim_{z\to 2}\frac{1}{z} = \frac{1}{2}, \tag{1.58}$$

now it is better. So, $z_0 = 0$ is a pole of order 1 while $z_1 = 2$ is a pole of order 2 and, therefore, the function is a meromorphic one.
Since $q(z)$ is a rational function and the denominator is a polynomial we expect this is a meromorphic function; the singularity points are the roots of the polynomial at the denominator, so

$$3z + 2z^2 + z^3 = 0 \Rightarrow z(z^2 + 2z + 3) = 0 \tag{1.59}$$

from which we find

$$z_0 = 0, z_1 = \frac{-2+\sqrt{4-12}}{2} = \frac{-2+i\sqrt{8}}{2}, z_2 = \frac{-2-\sqrt{4-12}}{2} = \frac{-2-i\sqrt{8}}{2}. \tag{1.60}$$

We can now rewrite $q(z)$ as

$$q(z) = \frac{3z^3 + 2z^2 + 2}{(z-z_0)(z-z_1)(z-z_2)}; \tag{1.61}$$

we immediately recognize that $z_0, z_1$ and $z_2$ are all poles of degree 1. The possibility to decompose a polynomial into multiplication of monomials containing the roots of the polynomial itself is granted by the fundamental theorem of algebra, according to which, any polynomial of degree $n$ admits exactly $n$ roots (with multiplicity) only in a algebraic closed field (in this case $\mathbb{C}$) which means that any polynomial of degree $n \geq 1$ admits at least one root in the field. It is interesting to note that, despite this possibility, only solutions of polynomial with degree $n \leq 4$ can always be write down using radicals. This is a theorem based on Galois theory and permutation groups.
It is the turn of $\Gamma(z)$, this is a very important function called gamma Euler function. First let us massage the function

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(\frac{n+z}{n}\right)^{-1}e^{\frac{z}{n}} = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(\frac{n}{n+z}\right)e^{\frac{z}{n}}; \tag{1.62}$$

this an analytic function in $\mathbb{C}\setminus\mathbb{Z}_-$ but has singularities in $z \in \mathbb{Z}_-$. Seems evident that these singularities are polar ones and the gamma function is meromorphic, indeed

$$\lim_{z\to 0} m(z;0;1)\Gamma(z) = \lim_{z\to 0} z\frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(\frac{n}{n+z}\right)e^{\frac{z}{n}} = \prod_{n=1}^{\infty} 1 = 1 \tag{1.63}$$

and

$$\lim_{z\to -n^*} m(z;-n^*;1)\Gamma(z) = \lim_{z\to -n^*}(z+n^*)\frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(\frac{n}{n+z}\right)e^{\frac{z}{n}} = \prod_{n=1}^{\infty} 1 =$$
$$= -\frac{e^{\gamma n^*}}{n^*}\prod_{n=1}^{n^*-1}\left(\frac{n}{n-n^*}\right)e^{-\frac{n^*}{n}}\prod_{n=n*+1}^{\infty}\left(\frac{n}{n-n^*}\right)e^{-\frac{n^*}{n}}\lim_{z\to -n^*}(z+n^*)\left(\frac{n^*}{n^*+z}\right)e^{\frac{z}{n^*}} =$$
$$= -\frac{e^{\gamma n^*}}{n^*}\prod_{n=1}^{n^*-1}\left(\frac{n}{n-n^*}\right)e^{-\frac{n^*}{n}}\prod_{n=n*+1}^{\infty}\left(\frac{n}{n-n^*}\right)e^{-\frac{n^*}{n}}(n^*e^{-1}) = \frac{(-1)^{n^*}}{n^*!}. \tag{1.64}$$

This function plays a crucial role in the renormalization techniques of Quantum Field Theory (specially in dimensional regulazization where infinites are replaced by poles of Euler's gamma function).

# 2 Complex integration

## 2.1 Complex integral and line integrals

Consider the complex function $f(z) = sin(z)e^z$ and perform its integration along a snapped line from $(0,0)$ to $(2\pi, i\pi)$ which pass through $(0, i\pi)$ using the definition of complex integral.

The very definition would be using integral sums, however we know that these converges to Riemann line integrals, and we have

$$\begin{aligned} I = \int_\gamma f(z)dz = \int_\gamma [u(x,y)+iv(x,y)][dx+idy] = \\ = \int_\gamma u(x,y)dx - v(x,y)dy + i\int_\gamma u(x,y)dy + v(x,y)dx. \end{aligned} \tag{2.1}$$

So we need to find real and imaginary parts, let us start with $sin(z)$

$$sin(z) = sin(x+iy) = sin(x)con(iy)+cos(x)sin(iy) = sin(x)cosh(y)+icos(x)sinh(y), \tag{2.2}$$

while

$$e^z = e^{x+iy} = e^x[cos(y)+isin(y)]; \tag{2.3}$$

therefore we have

$$\begin{aligned} u(x,y) = sin(x)cosh(y)e^x cos(y) - cos(x)sinh(y)e^x sin(y); \\ v(x,y) = sin(x)cosh(y)e^x sin(y) + cos(x)sinh(y)e^x cos(y). \end{aligned} \tag{2.4}$$

The integration path is given by two straight lines, one from (0,0) to $(0, i\pi)$ and the other one from $(0, i\pi)$ to $(2\pi, i\pi)$; so along the first line only $y$ varies while along the second one only $x$ varies. using property of integral we can split it into integrals along the two lines $I = I_{\gamma_1} + I_{\gamma_2}$; therefore for the first line $\gamma_1$

$$\begin{aligned} I_{\gamma_1} &= -\int_{\gamma_1} v(0,y)dy + i\int_{\gamma_1} u(0,y)dy = \\ &= -\int_0^{\pi} cos(y)sinh(y)dy - i\int_0^{\pi} sin(y)sinh(y)dy \end{aligned} \tag{2.5}$$

while for the second line $\gamma_2$

$$\begin{aligned} I_{\gamma_2} &= \int_{\gamma_2} u(x,\pi)dx + i\int_{\gamma_2} v(x,\pi)dx = \\ &= -\int_0^{2\pi} sin(x)cosh(\pi)e^x dx - i\int_0^{2\pi} cos(x)sinh(\pi)e^x dx. \end{aligned} \tag{2.6}$$

These integrals are solved using repeated integration by parts $\int f dg = fg - \int g df$, let us show how works in the case

$$\int cos(y)sinh(y)dy. \tag{2.7}$$

Take $f = cos(y)$ and $dg = sinh(y)dy$, therefore $df = -sin(y)dy$ and $g = cosh(y)$ and

$$\int cos(y)sinh(y)dy = cos(y)cosh(y) + \int cosh(y)sin(y)dy; \tag{2.8}$$

now take $f = sin(y)$ and $dg = cosh(y)dy$, therefore $df = cos(y)dy$ and $g = sinh(y)$, so

$$\int cosh(y)sin(y)dy = sin(y)sinh(y) - \int sinh(y)cos(y)dy. \tag{2.9}$$

To sum up, we have

$$2\int sinh(y)cos(y)dy = cos(y)cosh(y) + sin(y)sinh(y); \tag{2.10}$$

in the end

$$\int sinh(y)cos(y)dy = \frac{1}{2}[cos(y)cosh(y) + sin(y)sinh(y)]. \tag{2.11}$$

The other integrals are very similar, the result is

$$I_{\gamma_1} = \frac{1}{2}(1 + cosh(\pi)) - \frac{i}{2}sinh(\pi), \tag{2.12}$$

and

$$I_{\gamma_2} = \frac{cosh(\pi)}{2}(e^{2\pi} - 1) + \frac{isinh(\pi)}{2}(1 - e^{2\pi}). \tag{2.13}$$

Finally

$$\begin{aligned} I &= \frac{1}{2}(1 + cosh(\pi)) - \frac{i}{2}sinh(\pi) + \frac{cosh(\pi)}{2}(e^{2\pi} - 1) + \frac{isinh(\pi)}{2}(1 - e^{2\pi}) = \\ &= \frac{1}{2}(1 + e^{2\pi}) - ie^{2\pi}. \end{aligned} \tag{2.14}$$

## 2.2 Cauchy theorem and Morera theorem

Given the following complex functions:

1. $f(z) = \frac{e^z}{z-8}$;
2. $g(z) = sin(z)cos(z)$;
3. $h(z) = e^{cos(z)}$;
4. $\Gamma(z) = \int_0^\infty s^{z-1}e^{-s}ds \quad \text{for} \quad Re(z) > 1$;
5. $\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z} \quad \text{for} \quad Re(z) > 1$;

say if they are analytic and compute their integrals along the curve $\gamma := \{z \in \mathbb{C} \mid |z-2| = 1\}$.

Function $f(z)$ is the product of $e^z$ and $\frac{1}{z-8}$; $e^z$ is an analytic function while $\frac{1}{z-8}$ has a polar singularity in $z_0 = 8$, however the curve $\gamma$ is a circumference of radius $R = 1$ and center $z_0 = 2$ and in the domain $\mathbb{D}$ with frontiers $\partial\mathbb{D} = \gamma$ the function $f(z)$ is analytical, therefore by Cauchy theorem we get

$$\int_\gamma f(z)dz = 0. \tag{2.15}$$

Functions $g(z)$ and $h(z)$ are even more simpler cases. Indeed they are just combination or composition of analytic functions, so they are analytic; therefore by Cauchy theorem their integrals along every closed curve in the complex plane (including $\gamma$) is vanishing. We now consider more interesting cases. Let us start with the gamma function; first we would prove that this is analytical but we can do more, we can show at once that this function in analytical and its integral along every closed line is zero. Let us consider

$$\begin{aligned} \int_C \Gamma(z)dz = \int_C \int_0^\infty s^{z-1}e^{-s}dsdz = \int_0^\infty \left( \int_C s^{z-1}e^{-s}dz \right) ds = \\ = \int_0^\infty e^{-s} \left( \int_C s^{z-1}dz \right) ds, \end{aligned} \tag{2.16}$$

since $s^{z-1}$ is surely an analytic function in the domain of integration and inside it, the integral $\int_C s^{z-1}dz = 0$ due to Cauchy theorem; we have got that

$$\int_C \Gamma(z)dz = 0 \tag{2.17}$$

for every closed curve, therefore thanks to Morera theorem (that is, if a continuous, complex-valued function defined on an open set $\mathbb{D}$ in the complex plane has vanishing integral along any closed curve in $\mathbb{D}$, the function must be analytical) we can conclude that $\Gamma(z)$ is an analytical function. Similar considerations hold for $\zeta(z)$ called Riemann $\zeta$ function, indeed

$$\int_C \zeta(z)dz = \int_C \sum_{n=1}^{\infty} \frac{1}{n^z} dz = \sum_{n=1}^{\infty} \int_C \frac{1}{n^z} = 0, \tag{2.18}$$

since $\frac{1}{n^z}$ is an analytic function; again thanks to Morera theorem we conclude that Riemann $\zeta$ function is analytic. This function plays a crucial role in mathematics due to its important connection with the theory of prime numbers.

## 2.3 Principal value integrals, circumference arcs and complex integration

Compute the integrals

- $I = \int_{-ia}^{+ib} g(z)dz = \int_{-ia}^{+ib} \frac{1}{z^3} dz$;
- $II = \lim_{\rho\to\infty} \int_{\mathbb{D}:=\left\{|z|=\rho \ \mid \ arg(z)\in\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right\}} h(z) = \lim_{\rho\to\infty} \int_{\mathbb{D}:=\left\{|z|=\rho \ \mid \ arg(z)\in\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right\}} e^{-z}dz$;
- $III = \lim_{\rho\to\infty} \int_{C^+_{\frac{1}{2}\rho}} p(z)dz = \lim_{\rho\to\infty} \int_{C^+_{\frac{1}{2}\rho}} e^{i3z}\left(\frac{z^3}{4(z^3-1)} - \frac{1}{4}\right)dz$;
- $IV = \int_{\mathbb{Q}} \frac{q(z)}{z^2} dz = \int_{\mathbb{Q}} \frac{e^z}{z^2\sqrt{3+z}} dz$ with $\mathbb{Q}$ is the square of vertex $1, i, -1, -i$;
- $V = \int_{C^+_{\frac{1}{2}\rho=2}+[-2,2]} l(z)dz = \int_{C^+_{\frac{1}{2}\rho=2}+[-2,2]} \frac{z}{\bar{z}} dz$ (clockwise);
- $VI = \int_{[0,i]+[i,1+i]+[0,1+i]} j(z)dz = \int_{[0,i]+[i,1+i]+[0,1+i]} (Im(z) - Re(z) - 3i(Re(z)^2))dz$;
- $VII = \int_{C_{\rho=1}} G(z)dz = \int_{C_{\rho=1}} \frac{log(z)}{z} dz$ (counterclockwise)
- $VIII = \int_{C_{\rho=2}} H(z)dz = \int_{C_{\rho=2}} z^{\frac{3}{2}} + z^{\frac{4}{3}} + z^z(log(z)+1)dz$ (counterclockwise)

Function $g(z)$ has a polar singularity in $z = 0$ therefore we need to split the integral in the neighborhood of the singular point, moreover we change variable, $z = iy$, since the path is along the imaginary axis; we get

$$\begin{aligned} I &= \lim_{\epsilon\to 0}\left(\int_{-a}^{0-\epsilon} \frac{1}{(iy)^3} idy\right) + \lim_{\epsilon\to 0}\left(\int_{0+\epsilon}^{+b} \frac{1}{(iy)^3} idy\right) = \\ &= \lim_{\epsilon\to 0}\left(-\int_{-a}^{0-\epsilon} \frac{dy}{y^3}\right) + \lim_{\epsilon\to 0}\left(-\int_{0+\epsilon}^{+b} \frac{dy}{y^3}\right) = \\ &= \lim_{\epsilon\to 0}\left(\frac{1}{2y^2}\bigg|_{-a}^{0-\epsilon}\right) + \lim_{\epsilon\to 0}\left(\frac{1}{2y^2}\bigg|_{0+\epsilon}^{+b}\right) = \\ &= \lim_{\epsilon\to 0}\left(\frac{1}{2\epsilon^2} - \frac{1}{2a^2}\right) + \lim_{\epsilon\to 0}\left(\frac{1}{2b^2} - \frac{1}{2\epsilon^2}\right), \end{aligned} \tag{2.19}$$

which is not define, therefore we need to use the Cauchy's principal value

$$PV(I) = \lim_{\epsilon\to 0}\left(\frac{1}{2\epsilon^2} - \frac{1}{2a^2} + \frac{1}{2b^2} - \frac{1}{2\epsilon^2}\right) = \frac{1}{2b^2} - \frac{1}{2a^2}. \tag{2.20}$$

Let us consider the case $II$, to apply whatever of the lemmas for infinite arcs we need to show that $zh(z) \to_{\rho\to\infty} 0$ or to a constant $c$ uniformly. Therefore we need to show that $zf(z) \to_{\rho\to\infty} 0$ uniformly

$$0 \le |zf(z)| = \rho|e^{-z}| = \rho|e^{-\rho e^{i\theta}}| = \rho|e^{-\rho[cos(\theta)+isen(\theta)]}| = \rho|e^{-\rho cos(\theta)}e^{-isen(\theta)}| = \rho e^{-\rho cos(\theta)}, \tag{2.21}$$

this function converge uniformly to zero in the I and IV quadrants (where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that $0 \le cos(\theta) \le 1$) since we have

$$\rho e^{-\rho cos(\theta)} \le \rho e^{-\rho} \to_{\rho\to\infty} 0. \tag{2.22}$$

Since the integration path is the circular sector from 30° to 60°, the integral gives zero. If the function converged to a constant $c$ instead of zero the integral would give $ic\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = ic\frac{\pi}{6}$. Case $III$ seems easy, we use Jordan's lemma and we can conclude that the integral is zero since $3 > 0$. That is ok, but we need to check that the function $p(z)$ converge uniformly to zero. Let us compute the modulus of $\frac{z^3}{4(z^3-1)}$,

$$0 \le \left|\frac{z^3}{4(z^3-1)}\right| = \left|\frac{\rho^3 e^{3i\theta}}{4(\rho^3 e^{3\theta}-1)}\right| = \frac{\rho^3}{4|(\rho^3 e^{3\theta}-1)|} \le \frac{\rho^3}{4|\rho^3-1|} \to_{\rho\to\infty} \frac{1}{4}\ uniformly; \tag{2.23}$$

the last inequality is due to the fact that $|u - v| \ge ||u| - |v||$. Therefore the function $p(z) \to_{\rho\to\infty} 0$ uniformly and the Jordan's lemma can be applied. Let us look case $IV$, function $q(z) = e^z(3+z)^{-\frac{1}{2}} = e^{z-\frac{1}{2}log(3+z)}$ is analytic in $\mathbb{C}\setminus\mathbb{D}$ where $\mathbb{D} = -3 - t$ with $t \in [0,\infty)$ is the branch cut of the principal branch of the logarithm. Therefore we can use the Cauchy formula (recall that complex integrals are well defined also if the function is not defined or not continuous in a finite number of point along the path)

$$q^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\mathbb{Q}} \frac{q(w)}{(w-z)^{n+1}dw} \tag{2.24}$$

where $w = 0$ and $n = 1$, therefore we have

$$\int_{\mathbb{Q}} \frac{e^z}{z^2\sqrt{3+z}} = \frac{2\pi i}{1!}\left(\frac{e^z}{\sqrt{3+z}}\right)'\Bigg|_{z=0} = 2\pi i\left(1 - \frac{1}{2(3+z)}\right)e^{z-\frac{1}{2}log(3+z)}\Bigg|_{z=0} = \frac{5\pi i}{3\sqrt{3}}. \tag{2.25}$$

Case $V$ is simply and since $l(z)$ is not analytical we expect its integral along the path to be different from zero. Let us split the integral as

$$\int_{C^+_{\frac{1}{2}\rho=2}+[-2,2]} l(z)dz = \int_{C^+_{\frac{1}{2}\rho=2}} l(z)dz + \int_{[-2,2]} l(z)dz, \tag{2.26}$$

and let us parametrize the paths as

$$\begin{aligned} &C^+_{\frac{1}{2}\rho=2} = 2e^{i\theta}, \quad with\ 0 \leq \theta \leq \pi; \\ &[-2,2] = t \quad with - 2 \leq t \leq 2. \end{aligned} \tag{2.27}$$

We use the definition

$$\int_\gamma l(z)dz = -\int_a^b l(\gamma(t))\gamma'(t)dt; \tag{2.28}$$

where the minus is due to the orientation of the integation path. Using $dz = d(2e^{i\theta}) = 2e^{i\theta}id\theta$ we have

$$\int_{C^+_{\frac{1}{2}\rho=2}} l(z)dz = \int_0^\pi \frac{2e^{i\theta}}{2e^{-i\theta}}2e^{i\theta}id\theta = 2i\int_0^\pi e^{3i\theta}d\theta = \frac{2i}{3i}e^{3i\theta}\bigg|_0^\pi = \frac{2}{3(-1-1)} = -\frac{4}{3}; \tag{2.29}$$

and

$$\int_{[-2,2]} l(z)dz = \int_{-2}^2 \frac{t}{t}dt = t\bigg|_{-2}^2 = 4. \tag{2.30}$$

In the end we have got

$$\int_{C^+_{\frac{1}{2}\rho=2}+[-2,2]} l(z)dz = -\frac{4}{3} + 4 = \frac{8}{3}. \tag{2.31}$$

It is now time of case $VI$, and again, since $j(z)$ is not analytical we expect its integral along the path to be different from zero; let us split again the integral as

$$\int_{[0,i]+[i,1+i]+[0,1+i]} j(z)dz = \int_{[0,i]} j(z)dz + \int_{[i,1+i]} j(z)dz + \int_{[0,1+i]} j(z)dz, \tag{2.32}$$

and let us parametrize the paths as

$$\begin{aligned} &[0,i] = iy \quad with\ 0 \leq y \leq 1; \\ &[i,1+i] = x + i \quad with\ 0 \leq x \leq 1; \\ &[0,1+i] = t + it \quad with\ 0 \leq t \leq 1; \end{aligned} \tag{2.33}$$

Using again the definition we have

$$\begin{aligned} &\int_{[0,i]} j(z)dz = \int_0^1 y1idy = i\frac{x^2}{2}\bigg|_0^1 = \frac{i}{2}; \\ &\int_{[i,1+i]} j(z)dz = \int_0^1 (1 - x - 3ix^2)dx = \left(x - \frac{x^2}{2} - 3i\frac{x^3}{3}\right)\bigg|_0^1 = 1 - \frac{1}{2} - i = \frac{1}{2} - i; \\ &\int_{[0,1+i]} j(z)dz = \int_0^1 (t - t - 3it^2)(1+i)dt = -3i(1+i)\left(\frac{t^3}{3}\right)\bigg|_0^1 = -\frac{3i(1+i)}{3} = 1 - i. \end{aligned} \tag{2.34}$$

In the end

$$\int_{[0,i]+[i,1+i]+[0,1+i]} j(z)dz = \frac{i}{2} + \frac{1}{2} - i + 1 - i = \frac{3}{2} - \frac{3}{2}i. \tag{2.35}$$

Consider now case $VII$. The function is polydrome and we have to choose a branch cut; however this is not so important since the cut intersect just one point of the integration path and so the integral is well defined. The only thing we have to care is to choose the branch consistently with the parametrization of the path. For example we can choose the branch with $0 < arg(z) < 2\pi$ and the integration parameter as $0 \leq \theta \leq 2\pi$. The integral gives (using the definition and that $ln(z) = ln(\rho e^{i\theta}) = ln(1) + i\theta$)

$$\begin{aligned} \int_{C_{\rho=1}} G(z)dz &= \int_0^{2\pi} e^{-i\theta}(ln(1) + i\theta)e^{i\theta} i d\theta = \int_0^{2\pi} i^2\theta d\theta = \\ &= -\int_0^{2\pi} \theta d\theta = -\frac{\theta^2}{2}\bigg|_0^{2\pi} = -2\pi^2. \end{aligned} \tag{2.36}$$

The last case can be divided in three integrals; let $z = re^{i\theta}$ therefore we have

$$\begin{aligned} &z^{\frac{3}{2}} = r^{\frac{3}{2}} e^{i\frac{3}{2}\theta}; \\ &z^{\frac{4}{3}} = r^{\frac{4}{3}} e^{i\frac{4}{3}\theta}; \\ &z^z(log(z) + 1) = z^{zlog(z)}(log(z) + 1); \end{aligned} \tag{2.37}$$

all this function are polydrome and we have to choose a branch. Let us choose the principal one, $-\pi < \theta < \pi$; therefore we have a branch cut in the negative real axis and the curve must be modified into a circumference $C^{\epsilon}_{\rho=2}$ with parameter given by $-\pi + \epsilon \leq \theta \leq \pi - \epsilon$ in the limit $\epsilon \to 0$. In this region and on $C^{\epsilon}_{\rho=2}$, our functions are well defined and continuous, therefore they admits a primitive and we can see that

$$\begin{aligned} &\left(\frac{2}{5} z^{\frac{5}{2}}\right)' = z^{\frac{3}{2}}; \\ &\left(\frac{3}{7} z^{\frac{7}{3}}\right)' = z^{\frac{4}{3}}; \\ &(z^z)' = z^z(log(z) + 1). \end{aligned} \tag{2.38}$$

So we have

$$\begin{aligned}\int_{C_{\rho=2}} H(z)dz &= \int_{C_{\rho=2}} z^{\frac{3}{2}}dz + \int_{C_{\rho=2}} z^{\frac{4}{3}}dz + \int_{C_{\rho=2}} z^z(log(z)+1)dz = \\ &= \lim_{\epsilon\to 0}\left(\int_{C^{\epsilon}_{\rho=2}} z^{\frac{3}{2}}dz + \int_{C^{\epsilon}_{\rho=2}} z^{\frac{4}{3}}dz + \int_{C^{\epsilon}_{\rho=2}} z^z(log(z)+1)dz\right) = \\ &= \lim_{\epsilon\to 0}\left(\frac{2}{5}z^{\frac{5}{2}}\Big|_{z=2e^{i(-\pi+\epsilon)}}^{z=2e^{i(\pi-\epsilon)}} + \frac{3}{7}z^{\frac{7}{3}}\Big|_{z=2e^{i(-\pi+\epsilon)}}^{z=2e^{i(\pi-\epsilon)}} + z^z\Big|_{z=2e^{i(-\pi+\epsilon)}}^{z=2e^{i(\pi-\epsilon)}}\right) = \\ &= \lim_{\epsilon\to 0}\left(\frac{2}{5}2^{\frac{5}{2}}\left(e^{i\frac{5}{2}(\pi-\epsilon)} - e^{i\frac{5}{2}(-\pi+\epsilon)}\right)\right) + \\ &+ \lim_{\epsilon\to 0}\left(\frac{3}{7}2^{\frac{7}{3}}\left(e^{i\frac{7}{3}(\pi-\epsilon)} - e^{i\frac{7}{3}(-\pi+\epsilon)}\right)\right) + \\ &+ \lim_{\epsilon\to 0}\left(e^{2e^{i(\pi-\epsilon)}(log(2)+i(\pi-\epsilon))} - e^{2e^{i(-\pi+\epsilon)}(log(2)+i(-\pi+\epsilon))}\right)\end{aligned} \tag{2.39}$$

where we have used the limit of $\epsilon \to 0$ since we have to be careful on the branch cut. Now we need some algebra:

$$\begin{aligned}&\lim_{\epsilon\to 0}\left(\frac{2}{5}2^{\frac{5}{2}}\left(e^{i\frac{5}{2}(\pi-\epsilon)} - e^{i\frac{5}{2}(-\pi+\epsilon)}\right)\right) = \lim_{\epsilon\to 0}\left(\frac{4i}{5}2^{\frac{5}{2}}sin\left(\frac{5}{2}(\pi-\epsilon)\right)\right) = \\ &= \frac{4i}{5}2^{\frac{5}{2}}sin\left(\frac{5}{2}\pi\right) = \frac{4i}{5}2^{\frac{5}{2}};\end{aligned} \tag{2.40}$$

$$\begin{aligned}&\lim_{\epsilon\to 0}\left(\frac{3}{7}2^{\frac{7}{3}}\left(e^{i\frac{7}{3}(\pi-\epsilon)} - e^{i\frac{7}{3}(-\pi+\epsilon)}\right)\right) = \lim_{\epsilon\to 0}\left(\frac{6i}{7}2^{\frac{7}{3}}sin\left(\frac{7}{3}(\pi-\epsilon)\right)\right) = \\ &= \frac{6i}{7}2^{\frac{7}{3}}sin\left(\frac{7}{3}\pi\right) = \frac{3i\sqrt{3}}{7}2^{\frac{7}{3}};\end{aligned} \tag{2.41}$$

$$\begin{aligned}&\lim_{\epsilon\to 0}\left(e^{2e^{i(\pi-\epsilon)}(log(2)+i(\pi-\epsilon))} - e^{2e^{i(-\pi+\epsilon)}(log(2)+i(-\pi+\epsilon))}\right) = e^{2e^{i\pi}(log(2)+i\pi)} - e^{2e^{-i\pi}(log(2)-i\pi)} = \\ &= e^{-2(log(2)+i\pi)} - e^{-2(log(2)-i\pi)} = \underbrace{e^{-2log(2)}}_{=\frac{1}{4}}[e^{-2i\pi} - e^{2i\pi}] = -\frac{2isin(2\pi)}{4} = 0;\end{aligned} \tag{2.42}$$

finally

$$\int_{C_{\rho=2}} H(z)dz = \frac{4i}{5}2^{\frac{5}{2}} + \frac{3i\sqrt{3}}{7}2^{\frac{7}{3}}. \tag{2.43}$$

# 3 Series and residues

## 3.1 Taylor series

Assuming for the polyhydrome functions the principal branch, compute the Taylor expansion of the following functions:

- $f(z) = \int_0^z e^{w^2} dw$ for $z_0 = 0$;
- $g(z) = (z+1)log(1+z^2)$ for $z_0 = 0$;
- $h(z) = log\left(\frac{1+z}{1-z}\right)$ for $z_0 = 0$;
- $p(z) = log(z^2)$ for $z_0 = 1, -1$;
- $q(z) = \frac{1}{sin(z)}$ for $z_0 = 0$.

For the first case we have $e^{w^2}$; we can expand it in Taylor series using the expansion for the exponential and changing $w \to w^2$

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!} \Rightarrow e^{w^2} = \sum_{n=0}^{\infty} \frac{w^{2n}}{n!}. \tag{3.1}$$

The radius of convergence of the series is all $\mathbb{C}$ since $e^{w^2}$ is an entire function, moreover the integration path is contained in the convergence domain of the series and the function is continuous (it is analytical) on this path; so we can exchange the integral and the sum. We get

$$f(z) = \sum_{n=0}^{\infty} \int_0^z \frac{w^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{w^{2n+1}}{2n+1}\bigg|_0^z = \frac{z^{2n+1}}{(2n+1)n!}. \tag{3.2}$$

Function $g(z)$ is not analytical and the Taylor expansion it is possible only in the analytical domain, so let us find it. As usual we impose the equation (for the principal branch)

$$1 + z^2 = -t, \quad t \geq 0, \tag{3.3}$$

whose solutions are

$$z = \pm\sqrt{-t-1} = \pm i\sqrt{1+t}, \quad t \geq 0. \tag{3.4}$$

The function is analytical in $\mathbb{C} \setminus (-\infty, -1] \cup [1, +\infty)$; therefore Taylor expansion exists in a circle of radius $R < 1$ centered in $z_0 = 0$. Using the Taylor expansion for the logarithm we get

$$g(z) = (z+1)\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^{2n+1} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^{2n}. \tag{3.5}$$

Let us now study function $h(z)$, again this is analytical in $\mathbb{C}$ minus the se given by the solution of

$$\frac{1+z}{1-z} = -t \quad t \geq 0 \Rightarrow 1 + z = -t + zt \Rightarrow z - zt = -t - 1 \Rightarrow z = \frac{1+t}{t-1} \quad t \geq 0; \tag{3.6}$$

this equation represents the real semiaxis $(-\infty, -1]$ if $t \in (0, 1^-)$ and $[1, +\infty)$ if $t \in (1^+, \infty)$; the function is analytic in $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$. Therefore Taylor expansion is possible only in $z_0 = 0$ and with convergence radius $R = 1$; in this region we have

$$\frac{d}{dz}log\left(\frac{1+z}{1-z}\right) = \frac{1-z}{1+z}\frac{(1-z+1+z)}{(1-z)^2} = \frac{2}{(1+z)(1-z)} = \frac{2}{1-z^2} = 2\sum_{n=0}^{\infty} z^{2n}; \tag{3.7}$$

where we used the geometric series representation. So

$$\begin{aligned} log\left(\frac{1+z}{1-z}\right) &= log\left(\frac{1+z}{1-z}\right) - \underbrace{log\left(\frac{1+0}{1-0}\right)}_{=0} = \int_0^z \frac{d}{dw}log\left(\frac{1+w}{1-w}\right)dw = \\ &= 2\int_0^z \sum_{n=0}^{\infty} w^{2n} dw = 2\sum_{n=0}^{\infty} \frac{w^{2n+1}}{2n+1}\bigg|_0^z = 2\sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1}. \end{aligned} \tag{3.8}$$

The hypothesis under which we can exchange the integral and sum are obviously satisfied since the path we consider is in the ball $B_1(0)$ (otherwise the series we have got does not converge) and the function is analytical in this ball.
Consider now function $p(z)$, as usual we have to solve

$$z^2 = -t, \quad t \geq 0, \tag{3.9}$$

solutions are

$$z = \pm i\sqrt{t}, \quad t \geq 0 \tag{3.10}$$

which represent the whole imaginary axis and the function is analytic in $\mathbb{C} \setminus \{Re(z) = 0\}$. If we center the expansion in $z_0 = -1, 1$ the maximum radius of convergence is $R = 1$ (otherwise we hit the imaginary axis where the function is not analytic). Note that (in the analyticity domain)

$$\frac{d}{dz}log(z^2) = \frac{2}{z} = \begin{cases} \frac{-2}{-z+1-1} = -2\frac{1}{1-(z+1)} & useful\ if\ z+1<1 \\ \frac{2}{z+1-1} = 2\frac{1}{1+(z-1)} & useful\ if\ z-1<1 \end{cases} \tag{3.11}$$

we have for $|z+1| < 1$ (expansion centered in $z_0 = -1$)

$$\begin{aligned} log(z^2) - \underbrace{log((-1)^2)}_{=0} &= \int_{-1}^z \frac{2}{w}dw = -2\int_{-1}^z \frac{1}{1-\underbrace{(w+1)}_{<1}} = -2\int_{-1}^z \sum_{k=0}^{\infty}(w+1)^n dw = \\ &= -2\sum_{k=0}^{\infty}\int_{-1}^z (w+1)^k dw = -2\sum_{k=0}^{\infty}\frac{(z+1)^{k+1}}{k+1}; \end{aligned} \tag{3.12}$$

while for $|z-1|<1$ (expansion centered in $z_0=1$)

$$log(z^2) - \underbrace{log((1)^2)}_{=0} = \int_1^z \frac{2}{w}dw = 2\int_1^z \frac{1}{1+\underbrace{(w-1)}_{<1}} = 2\int_1^z \sum_{k=0}^{\infty}(-1)^k(w-1)^k dw =$$
$$= 2\sum_{k=0}^{\infty}(-1)^k \int_1^z (w-1)^k dw = 2\sum_{k=0}^{\infty}(-1)^k \frac{(z-1)^{k+1}}{k+1}. \tag{3.13}$$

The last function seems easy but we need to pay attention. Fist of all we note that the function is analytic in $\mathbb{C}\setminus\{z\in\mathbb{C}|z=k\pi, k\in\mathbb{Z}\}$, therefore the Taylor expansion in $z_0=0$ has radius $R=\pi$; in general the expansion around $z=z_0\neq k\pi$ with $k\in\mathbb{Z}$ has radius $R=\pi$. Let us use the finite expansion method:

$$\begin{aligned}
\frac{1}{sin(z)} &= \frac{1}{z-\frac{z^3}{3}+\frac{z^5}{5!}-\frac{z^7}{7!}+...} = \frac{1}{z}\frac{1}{1-(\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^6}{7!}+...)} = \\
&= \frac{1}{z}\left[1+\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^6}{7!}+...\right)+\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^6}{7!}+...\right)^2+...\right] = \\
&= \frac{1}{z}\left[1+\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^4}{3!3!}+\frac{z^6}{7!}-2\frac{z^6}{3!5!}+\frac{z^6}{3!3!}+...\right] = \\
&= \frac{1}{z}+\frac{1}{6}z+\frac{7}{360}z^3+...
\end{aligned} \tag{3.14}$$

note that we used $\frac{1}{1-h(z)}=\sum_0^\infty h^n(z)$ that holds if $h(z)<1$ which is satisfied in this kind of expansions.

## 3.2 Laurent series

Assuming for the polyhydrome functions the principal branch, compute the Laurent expansion of the following functions:

- $f(z)=\frac{1}{(z(1-z))^2}$ for $z_0=0$;
- $g(z)=\frac{3+z}{z^3+2z^2}$ for $z_0=0$;
- $p(z)=\left(\frac{1}{z^2sinh(z)}\right)'|_{z_0=0}$;
- $q(z)=\frac{1}{zsin(z)}$ for $z_0=0$.

Let us begin with $f(z)$; this function has singularities in $z=0$ and $z=1$ therefore the annular regions where the function is analytic are $A(0,0,1)$ and $A(0,1,\infty)$. In the first ring we have

$$\frac{1}{(z(1-z))^2} = \frac{1}{z^2}\frac{1}{(1-z)^2} = \frac{1}{z^2}\frac{d}{dz}\left(\frac{1}{1-\underbrace{z}_{<1}}\right) = \frac{1}{z^2}\frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n\right) = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{d}{dz}z^n =$$
$$= \frac{1}{z^2}\sum_{n=1}^{\infty} n z^{n-1} = \sum_{n=1}^{\infty} n z^{n-3}; \tag{3.15}$$

while in the second ring we have to rename the variable, indeed using $w = \frac{1}{z}$ we have that $|w| < 1$ (we are in the ring $A(0,1,\infty)$)

$$\frac{1}{(z(1-z))^2} = \frac{w^2}{(1-\frac{1}{w})^2} = \frac{w^2}{(\frac{w-1}{w})^2} = \frac{w^4}{(1-w)^2} = w^4\frac{d}{dw}\left(\frac{1}{1-\underbrace{w}_{<1}}\right) =$$
$$= w^4\frac{d}{dw}\left(\sum_{n=0}^{\infty} w^n\right) = w^4\sum_{n=0}^{\infty}\frac{d}{dw}w^n = w^4\sum_{n=1}^{\infty} n w^{n-1} = \sum_{n=1}^{\infty} n w^{n+3} =$$
$$= \sum_{n=1}^{\infty} n z^{3-n}. \tag{3.16}$$

Note that we can exchange the sums and the derivatives because we are in the analytic domain. Moreover, fundamental for this kind of exercises are the expansions

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad \frac{1}{1+z} = \sum_{n=0}^{\infty}(-1)^n z^n \qquad for |z| < 1 \tag{3.17}$$

Function $g(z)$ has singularities in $z = 0$ and $z = 2$ so we can expand in Laurent series in the rings $A(0,0,2)$ and $A(0,2,\infty)$. In the first ring we have

$$\frac{3+z}{z^3+2z^2} = \frac{3+z}{2z^2(1+\underbrace{\frac{z}{2}}_{<1})} = \frac{3+z}{2z^2}\frac{1}{1-\frac{z}{2}} = \frac{3+z}{2z^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n =$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{(3+z)}{2z^2}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{3}{2^{n+1}}z^{n-2} + \sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{n+1}}z^{n-1}; \tag{3.18}$$

while in the ring $A(0,2,\infty)$ we have

$$\frac{3+z}{z^3+2z^2} = \frac{3+z}{z^3(1+\underbrace{\frac{2}{z}}_{<1})} = \frac{3+z}{z^3}\frac{1}{1+\frac{2}{z}} = \frac{3+z}{z^3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n =$$
$$= \sum_{n=0}^{\infty}(-1)^n 3\frac{2^n}{z^{n+3}} + \sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^{n+2}}. \tag{3.19}$$

For function $p(z)$ we need to expand using the finite expansion method function $\frac{1}{z^2 sinh(z)}$; we have

$$\begin{aligned}\frac{1}{z^2 sinh(z)} &= \frac{1}{z^2}\frac{1}{z+\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}+...} = \frac{1}{z^3}\frac{1}{1+\frac{z^2}{3!}+\frac{z^4}{5!}+\frac{z^6}{7!}+...} = \\ &= \frac{1}{z^3}\left[1-\left(\frac{z^2}{3!}+\frac{z^4}{5!}+\frac{z^6}{7!}+...\right)+\left(\frac{z^2}{3!}+\frac{z^4}{5!}+\frac{z^6}{7!}+...\right)^2-...\right] = \\ &= \frac{1}{z^3}\left[1-\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^4}{3!3!}-\frac{z^6}{7!}+2\frac{z^6}{3!5!}+...\right] = \\ &= \frac{1}{z^3}-\frac{1}{6z}+\frac{7}{360}z+...\end{aligned} \tag{3.20}$$

and taking the derivative we get

$$\left(\frac{1}{z^2 sinh(z)}\right)' = -\frac{3}{z^4}-\frac{1}{3z^2}+\frac{7}{360}+... \tag{3.21}$$

For function $q(z)$ we can use the expansion find before,

$$\begin{aligned}\frac{1}{sin(z)} &= \frac{1}{z-\frac{z^3}{3}+\frac{z^5}{5!}-\frac{z^7}{7!}+...} = \frac{1}{z}\frac{1}{1-(\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^6}{7!}+...)} = \\ &= \frac{1}{z}\left[1+\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^6}{7!}+...\right)+\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^6}{7!}+...\right)^2+...\right] = \\ &= \frac{1}{z}\left[1+\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^4}{3!3!}+\frac{z^6}{7!}-2\frac{z^6}{3!5!}+...\right] = \\ &= \frac{1}{z}+\frac{1}{6}z+\frac{7}{360}z^3+...\end{aligned} \tag{3.22}$$

so we have

$$\begin{aligned}\frac{1}{z sin(z)} &= \frac{1}{z}\frac{1}{z-\frac{z^3}{3}+\frac{z^5}{5!}-\frac{z^7}{7!}+...} = \frac{1}{z}\frac{1}{z}\frac{1}{1-(\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^6}{7!}+...)} = \\ &= \frac{1}{z}\frac{1}{z}\left[1+\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^6}{7!}+...\right)+\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^6}{7!}+...\right)^2+...\right] = \\ &= \frac{1}{z}\frac{1}{z}\left[1+\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^4}{3!3!}+\frac{z^6}{7!}-2\frac{z^6}{3!5!}+...\right] = \\ &= \frac{1}{z^2}+\frac{1}{6}+\frac{7}{360}z^2+...\end{aligned} \tag{3.23}$$

## 3.3 Residues

Determine the nature of the singularities and compute, when possible, the residues of the following functions:

- a generic function $q(z)$ with a pole of order $m$;

- $f(z) = \frac{1}{(e^z-1)sin(z)}$;
- $l(z) = \frac{3+z}{z^3+2z^2}$;
- $g(z) = zlog(z)$;
- $h(z) = \frac{1}{log(1+z)}$;
- $p(z) = \frac{e^z}{z^2 sin(z)}$.

The first case is a useful formula to compute residues in the case of polar singularities. Suppose the function has a pole of order $m$ in $z = z_0$; there exist $\epsilon > 0$ such that the Laurent expansion exists (the negative part end with a monomial $(z - z_0)^{-m}$)

$$q(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + ... + \frac{b_m}{(z - z_0)^m}; \tag{3.24}$$

if we multiply both member by $(z - z_0)^m$ in order to isolate the coefficients $b_m$ we get

$$(z - z_0)^m q(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^{k+m} + b_1(z - z_0)^{m-1} + b_2(z - z_0)^{m-2} + ... + b_m. \tag{3.25}$$

We need to compute $b_1$ so we take $m - 1$ derivatives with respect to $z$ in order to remove the $z$-dependence in the term with $b_1$:

$$\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m q(z)] = \sum_{k=0}^{\infty} a_k(k+m)(k+m-1)...(k+2)(z-z_0)^{k+1} + b_1(m-1)(m-2)...1. \tag{3.26}$$

It is now easy to see that we need to multiply by $\frac{1}{(m-1)!}$ and to take the limit

$$b_1 = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m q(z)] \tag{3.27}$$

and this is the residue of a pole of order $m$.
Let us consider now function $f(z)$, we can Taylor expand

$$f(z) = \frac{1}{(z + \frac{z^2}{2} + \frac{z^3}{6} + ...)(z - \frac{z^3}{6} + ...)} = \frac{1}{z^2} \frac{1}{(1 + \frac{z}{2} + \frac{z^2}{6} + ...)(1 - \frac{z^2}{6} + ...)} = \frac{\tilde{f}(z)}{z^2}; \tag{3.28}$$

recall that if we have a function with a polar singularity of order $m$ we can always write

$$f(z) = \frac{\tilde{f}(z)}{(z - z_0)^m} \tag{3.29}$$

and

$$Res_{z=z_0} f(z) = \frac{\tilde{f}(z_0)^{(m-1)}}{(m-1)!}. \tag{3.30}$$

In our case, we have a pole in $z_0 = 0$ of order $m = 2$ and therefore

$$
\begin{aligned}
Res_{z=0}f(z) &= \tilde{f}(0)^{(1)} = \\
&= -\frac{\frac{1}{2}+\frac{z}{3}+...}{(1+\frac{z}{2}+\frac{z^2}{6}+...)^2(1-\frac{z^2}{6}+...)} - \frac{-\frac{z}{3}+...}{(1+\frac{z}{2}+\frac{z^2}{6}+...)(1-\frac{z^2}{6}+...)^2}\Bigg|_{z=0} = \\
&= -\frac{1}{2}
\end{aligned}
\tag{3.31}
$$

Function $l(z)$ is similar to the previous case, indeed for the singularity in $z = 0$ (doble pole)

$$
l(z) = \frac{3+z}{z^3+2z^2} = \frac{1}{z^2}\frac{3+z}{2+z} = \frac{\tilde{l}_0(z)}{z^2}, \tag{3.32}
$$

so

$$
Res_{z=0}l(z) = \tilde{l}_0(0)^{(1)} = \frac{2+z-3-z}{(2+z)^2}\Bigg|_{z=0} = -\frac{1}{4}; \tag{3.33}
$$

while for the singularity in $z = -2$ (simple pole)

$$
l(z) = \frac{3+z}{z^3+2z^2} = \frac{1}{2+z}\frac{3+z}{z^2}, \tag{3.34}
$$

so

$$
Res_{z=-2}l(z) = \lim_{z\to-2}(z+2)\frac{1}{2+z}\frac{3+z}{z^2} = \frac{1}{4}. \tag{3.35}
$$

Function $g(z)$ is tricky, indeed since $z = 0$ is not an isolated singularity there is no ring where a Laurent expansion is possible and no residue exists.
Function $h(z)$ can be expanded in Laurent series using the Taylor expansion $log(1+z) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^n$ that converge for $|z| < 1$; therefore

$$
\begin{aligned}
\frac{1}{log(1+z)} &= \frac{1}{z-\frac{z^2}{2}+\frac{z^3}{3}+...} = \frac{1}{z}\frac{1}{1+(-\frac{z}{2}+\frac{z^2}{3}+...)} = \\
&= \frac{1}{z}\left[1-\left(-\frac{z}{2}+\frac{z^2}{3}+...\right)+\left(-\frac{z}{2}+\frac{z^2}{3}+...\right)^2+...\right] = \\
&= \frac{1}{z}+\frac{1}{2}-\frac{z}{12}+\frac{z^2}{24}+...,
\end{aligned}
\tag{3.36}
$$

and so $z = 0$ is a simple pole and the residues (the coefficient of $z^{-1}$) is simply $Res_{z=0}\frac{1}{log(1+z)} = 1$.
Let us consider the last case. We recall that if we have a function $p(z) = \frac{p_1(z)}{p_2(z)}$ with $p_1(z)$ and $p_2(z)$ analytic in $z = z_0$ and if $p_1(z_0) \neq 0, p_2'(z_0) \neq 0$ then if $z_0$ is a simple pole we have

$$
Res_{z=z_0}p(z) = \frac{p_1(z_0)}{p_2'(z_0)}. \tag{3.37}
$$

In our case we have a series of simple poles in $z_n = n\pi$ with $n \in \mathbb{Z}\setminus\{0\}$ and a pole of order $m = 3$ in $z_0 = 0$. Therefore, since in our case we have

$$
p_1(z) = e^z, \qquad p_2(z) = z^2sin(z) \Rightarrow p_2'(z) = 2zsin(z) + z^2cos(z) \tag{3.38}
$$

and $p_1(z)$ and $p_2'(z)$ are analytical in $z = z_n$ we can compute the residue as

$$Res_{z=z_n} p(z) = \frac{e^{n\pi}}{n^2\pi^2(-1)^n} = (-1)^n \frac{e^{n\pi}}{n^2\pi^2}. \tag{3.39}$$

To compute the residue in $z_0 = 0$ we need to compute the Laurent expansion of the function; we have

$$\begin{aligned} p(z) &= \frac{1 + z + \frac{z^2}{2} + \frac{z^3}{6} + ...}{z^2(z - \frac{z^3}{6} + ...)} = \frac{1}{z^3}\frac{1 + z + \frac{z^2}{2} + \frac{z^3}{6} + ...}{1 - (\frac{z^2}{6} + ...)} = \\ &= \frac{1}{z^3}\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + ...\right)\left[1 + \left(\frac{z^2}{6} + ...\right) + \left(\frac{z^2}{6} + ...\right)^2 + ...\right] = \\ &= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{6z} + \frac{1}{6} + ... = \frac{1}{z^3} + \frac{1}{z^2} + \frac{2}{3z} + \frac{1}{6} + ... \end{aligned} \tag{3.40}$$

where the last ... stands for positive powers of $z$. In the end

$$Res_{z=0} p(z) = \frac{2}{3}. \tag{3.41}$$

# 4 Residues integral

## 4.1 Integration using residues theorem

Solve the following integrals using the residues theorem.

- $I = \int_0^{2\pi} \frac{d\theta}{1+acos(\theta)} \qquad -1 < a < 1;$
- $II = \int_{-\infty}^{+\infty} \frac{xdx}{(x^2+1)(x^2+2x+2)};$
- $III = PV \int_{-\infty}^{+\infty} \frac{xdx}{x^3-1};$
- $IV = \int_{-\infty}^{+\infty} \frac{e^{ax}dx}{cosh(x)} \qquad -1 < a < 1;$
- $V = \int_{-\infty}^{+\infty} \frac{xsin(x)}{x^2+1}.$

Let us consider the first integral, we know that

$$\begin{aligned} \int_0^{2\pi} f(cos(\theta), sin(\theta))d\theta &= \int_0^{2\pi} f\left(\frac{e^{i\theta} + e^{-i\theta}}{2}, \frac{e^{i\theta} - e^{-i\theta}}{2i}\right)\frac{ie^{i\theta}}{ie^{i\theta}}d\theta = \\ &= \int_0^{2\pi} f\left(\frac{\gamma(\theta) + \gamma(\theta)^{-1}}{2}, \frac{\gamma(\theta) + \gamma(\theta)^{-1}}{2i}\right)\frac{\gamma'(\theta)}{i\gamma(\theta)}d\theta = \\ &= \int_\gamma \left(\frac{z + z^{-1}}{2}, \frac{z + z^{-1}}{2i}\right)\frac{dz}{iz} \end{aligned} \tag{4.1}$$

where $\gamma(\theta) = e^{i\theta}$ with $0 \leq \theta \leq 2\pi$. Therefore,

$$\int_0^{2\pi} \frac{d\theta}{1 + acos(\theta)} = \int_\gamma \frac{1}{1 + a(\frac{z+z^{-1}}{2})} \frac{dz}{iz} = \frac{2}{ia} \int_\gamma \frac{1}{z^2 + \frac{2z}{a} + 1} dz; \tag{4.2}$$

now

$$z^2 + \frac{2z}{a} + 1 = 0 \Rightarrow z_\pm = \frac{-\frac{2}{a} \pm \sqrt{\frac{4}{a^2} - 4}}{2} = \frac{-\frac{2}{a} \pm \frac{2}{a}\sqrt{1-a^2}}{2} = \frac{-1 \pm \sqrt{1-a^2}}{a}, \tag{4.3}$$

and we note that $|z_-| > 1$ and $|z_+| < 1$. So only $z_+$ is inside the circumference $\gamma$ and this is a simple pole; therefore for residues theorem we have

$$\begin{aligned} \frac{2}{ia} \int_\gamma \frac{1}{z^2 + \frac{2z}{a} + 1} dz &= \frac{2}{ia} 2\pi i Res_{z=z_+} \frac{1}{z^2 + \frac{2z}{a} + 1} = \frac{4\pi}{a} Res_{z=z_+} \frac{1}{(z - z_+)(z - z_-)} = \\ &= \frac{4\pi}{a} \lim_{z \to z_+} (z - z_+) \frac{1}{(z - z_+)(z - z_-)} = \frac{4\pi}{a} \frac{1}{z_+ - z_-} = \frac{2\pi}{\sqrt{1-a^2}}. \end{aligned} \tag{4.4}$$

For case $II$ we can use directly residues integral since $f(z) = \frac{x}{(x^2+1)(x^2+2x+2)}$ has only polar singularities in $z^2 + 1 = 0 \Rightarrow z_\pm = \pm i$ and $z^2 + 2z + 2 = 0 \Rightarrow z'_\pm = \frac{-2\pm\sqrt{4-8}}{2} = -1 \pm i$ with residues given by

$$\begin{aligned} Res_{z=z_\pm} f(z) = lim_{z\to z_\pm} (z - z_\pm) &\frac{z}{(z - z_+)(z - z_-)(z^2 + 2z + 2)} = \\ &= \left. \frac{z}{(z - (\pm i))(z^2 + 2z + 2)} \right|_{z=z_\pm} = \\ &= \frac{\pm i}{\pm 2i(-1 \pm 2i + 2)} = \frac{1 \mp 2i}{10} \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} Res_{z=z'_\pm} f(z) = lim_{z\to z'_\pm} (z - z'_\pm) &\frac{z}{(z^2 + 1)(z - z'_-)(z - z'_+)} = \\ &= \left. \frac{z}{(z^2 + 1)(z - (-1 \mp i))} \right|_{z=z'_\pm} = \\ &= \frac{-1 \pm i}{(1 \mp 2i)(\pm 2i)} = \frac{-1 \pm 3i}{10}. \end{aligned} \tag{4.6}$$

Choosing as path of integration the semicircumference in the upper half plane with radius $R$, we enclose only $z_+$ and $z'_+$, so in the limit $R \to \infty$ we get

$$\int_{-\infty}^{+\infty} \frac{xdx}{(x^2 + 1)(x^2 + 2x + 2)} = 2\pi i \left( \frac{1 - 2i}{10} + \frac{-1 + 3i}{10} \right) = 2\pi i \frac{i}{10} = -\frac{\pi}{5}. \tag{4.7}$$

Note that for a rational function $f(z) = \frac{P_n(x)}{Q_m(x)}$ where $P_n(x)$ ans $Q_n(x)$ are polynomial of degree $n$ and $m$ (with $m \geq n + 1$) respectively we get that the integral on the semicircumference goes to zero when $R \to \infty$.

Let us now consider case $III$; the function has pole in

$$z^3 - 1 = 0 \Rightarrow z_k =^3 \sqrt{1} = e^{\frac{i2k\pi}{3}}, \quad k = 0, 1, 2; \tag{4.8}$$

we note that $z_0$ is a real pole (this is why the principal value). Let us choose the semicircumference in the upper half plane with radius $R$ plus a little circumference of radius $r$ around $z_0$; this path encloses the simple pole in $z = z_1$ whose residues is given by

$$Res_{z=z_1} f(z) = \frac{p(z_1)}{q'(z_1)} = \frac{z}{3z^2}\bigg|_{z=z_1} = \frac{e^{\frac{i2\pi}{3}}}{3e^{\frac{i4\pi}{3}}} \tag{4.9}$$

where we write $f(z) = \frac{p(z)}{q(z)}$ with $p(z) = z$ and $q(z) = z^3 - 1$. In the limit $R \to \infty$ the integral on the semicircumference in the upper half plane goes to zero (like in case $II$) while in the limit $r \to 0$ we have for a generic function $g(z)$ that

$$\lim_{r \to 0} \int_{\gamma_r^{\pm}} g(z) dz = \pm i\pi Res_{z=z'} g(z), \tag{4.10}$$

where $\gamma_r^{\pm}(\theta) = z' + re^{\pm i\theta}$ where $-\pi \leq \theta \leq 0$ and $z'$ is the simple pole of the function $g(z)$. In our case we have $\gamma_r^- = z_0 + e^{-i\theta}$ and so

$$\lim_{r \to 0} \int_{\gamma_r^-} f(z) dz = i\pi Res_{z=z_0} f(z) = i\pi \frac{p(z_0)}{q'(z_0)} = i\pi \frac{1}{3}. \tag{4.11}$$

in the end

$$PV \int_{-\infty}^{+\infty} \frac{x dx}{x^3 - 1} = 2\pi i Res_{z=z_1} f(z) - i\pi Res_{z=z_0} f(z) = \frac{i\pi}{3}\left[2e^{-\frac{2i\pi}{3}} - 1\right]. \tag{4.12}$$

Let us consider case $IV$. Function $f(z) = \frac{e^{az}}{cosh(z)}$ has polar singularity in

$$\begin{aligned} cosh(z) &= cosh(x + iy) = cosh(x)cosh(iy) + sinh(x)sinh(iy) = \\ &= cosh(x)cos(y) + isinh(x)sin(y) = 0 \Rightarrow y = k\frac{\pi}{2} \cup x = 0 \quad with\ k \in \mathbb{Z} \setminus \{0\}; \end{aligned} \tag{4.13}$$

let us call these points $z_k$. We consider as integration path the square given by

$$\begin{aligned} &\gamma_1(x) = x \quad -R \leq x \leq R; \\ &\gamma_2(y) = R + iy \quad 0 \leq y \leq \pi; \\ &\gamma_3(x) = x + i\pi \quad R \leq x \leq -R; \\ &\gamma_4(y) = -R + iy \quad \pi \leq y \leq 0. \end{aligned} \tag{4.14}$$

The integral on $\gamma_1(x)$ is

$$\int_{-R}^{R} \frac{e^{ax}}{cosh(x)} dx \tag{4.15}$$

that is our integral when $R \to \infty$; the integral on $\gamma_2(y)$ is

$$\int_0^{\pi} \frac{e^{a(R+iy)}}{cosh(R+iy)}dy, \tag{4.16}$$

so

$$\left| \int_0^{\pi} \frac{e^{a(R+iy)}}{cosh(R+iy)}dy \right| \leq \int_0^{\pi} \left| \frac{e^{a(R+iy)}}{cosh(R+iy)} \right| dy \leq \pi \frac{e^{aR}}{\frac{1}{2}(e^R+e^{-R})} \to 0, \tag{4.17}$$

for $R \to \infty$ and $a < 1$; moreover we used $cosh(R+i\pi) = -cosh(R)$. integral on $\gamma_3(x)$ is

$$\int_R^{-R} \frac{e^{a(x+i\pi)}}{cosh(x+i\pi)dx} = e^{ia\pi} \int_{-R}^{R} \frac{e^{ax}}{cosh(x)dx} \tag{4.18}$$

we used again $cosh(x+i\pi) = -cosh(x)$; this is our integral multiplied by $e^{ia\pi}$ when $R \to \infty$; the last integral is

$$\int_{\pi}^{0} \frac{e^{a(-R+iy)}}{cosh(-R+iy)}dy, \tag{4.19}$$

so

$$\left| -\int_0^{\pi} \frac{e^{a(-R+iy)}}{cosh(-R+iy)}dy \right| \leq \int_0^{\pi} \left| \frac{e^{a(-R+iy)}}{cosh(-R+iy)} \right| dy \leq \pi \frac{e^{-aR}}{\frac{1}{2}(e^R+e^{-R})} \to 0, \tag{4.20}$$

for $R \to \infty$ and $a > -1$. Therefore

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{cosh(x)}dx(1+e^{ia\pi}) = 2\pi i Res_{z=z_1} = 2\pi i \frac{e^{az}}{sinh(z)}\Big|_{z=i\frac{\pi}{2}} = 2\pi i \frac{e^{ia\frac{\pi}{2}}}{\underbrace{sinh(\frac{i\pi}{2})}_{=isin(\frac{\pi}{2})=i}} \tag{4.21}$$

in the end

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{cosh(x)}dx = \frac{2\pi e^{ia\frac{\pi}{2}}}{1+e^{ia\pi}}. \tag{4.22}$$

Let us consider the last case. Consider function $f(z) = \frac{ze^{iz}}{z^2+1}$, this has simple pole in $z = \pm i$, so consider the path given by the semicircumference in the upper half plane with radius $R$ plus the interval from $-R$ to $R$. This path encloses the pole in $z = i$ but for Jordan's lemma the integral on the semicircumference in the upper half plane is zero in the limit $R \to \infty$. Therefore

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{xsin(x)}{x^2+1} &= Im \int_{-\infty}^{+\infty} \frac{ze^{iz}}{z^2+1} = Im\left(2\pi i Res_{z=i} f(z)\right) = \\ &= Im\left(2\pi i \lim_{z \to i}(z-i)\frac{ze^{iz}}{(z-i)(z+i)}\right) = Im\left(2\pi i \frac{ie^{-1}}{2i}\right) = \frac{\pi}{e}. \end{aligned} \tag{4.23}$$

# 5 Asymptotic developments and integral estimates

## 5.1 Integration by parts

Compute the asymptotic development up to second order and estimates the rest for the following integrals:

1. $\int_0^{+\infty} e^{-xt}t^3 dt$;
2. $\int_2^5 e^{ixt} log(t) dt$

Let us consider the first integral; this is of Laplace type therefore we know that

$$\int_0^{+\infty} e^{-xt}t^3 dt = \sum_{k=0}^{n-1} \frac{(t^2)^{(k)}(0)}{x^{k+1}} + \frac{1}{x^k}\int_0^{\infty} e^{-xt}(t^3)^{(n)}(t)dt \tag{5.1}$$

where the last integral it the rest. In our case we have

$$\int_0^{+\infty} e^{-xt}t^3 dt \sim \sum_{k=0}^{1} \frac{(t^2)^{(k)}(0)}{x^{k+1}} + \frac{1}{x^n}\int_0^{\infty} e^{-xt}(t^2)^{(2)}(t)dt; \tag{5.2}$$

we have $(t^2)^{(0)}(t) = t^2$, $(t^2)^{(1)}(t) = 2t$, $(t^2)^{(2)}(t) = 2$ so

$$\int_0^{+\infty} e^{ixt}t^3 dt \sim \frac{2}{x^2}\int_0^{\infty} e^{-xt}dt = -\frac{2}{x^2}\frac{1}{x}e^{-xt}\Big|_0^{+\infty} = \frac{2}{x^3}. \tag{5.3}$$

The second case in instead of Fourier type, therefore

$$\begin{aligned}&\int_2^5 e^{ixt}log(t)dt = \\ &= \sum_{k=0}^{n-1} \frac{i^{k+1}}{x^k+1}[e^{2ix}(log(t))^{(k)}(a) - e^{5ix}(log(t))^{(k)}(b)] + \frac{i^n}{x^n}\int_2^5 e^{ixt}(log(t))^{(n)}(t)dt\end{aligned} \tag{5.4}$$

In our case

$$\begin{aligned}&\int_2^5 e^{ixt}log(t)dt \sim \\ &\sim \sum_{k=0}^{1} \frac{i^{k+1}}{x^k+1}[e^{2ix}(log(t))^{(k)}(a) - e^{5ix}(log(t))^{(k)}(b)] - \frac{1}{x^2}\int_2^5 e^{ixt}(log(t))^{(2)}(t)dt,\end{aligned} \tag{5.5}$$

and $(log(t))^{(0)}(t) = log(t)$, $(log(t))^{(1)}(t) = \frac{1}{t}$, $(log(t))^{(2)}(t) = -\frac{1}{t^2}$; therefore

$$\begin{aligned}&\int_2^5 e^{ixt}log(t)dt \sim \\ &\sim \frac{i}{x}[e^{2ix}log(2) - e^{5ix}log(5)] - \frac{1}{x^2}[e^{2ix}\frac{1}{2} - e^{5ix}\frac{1}{5}] + \frac{1}{x^2}\int_2^5 e^{ixt}\frac{1}{t^2}dt.\end{aligned} \tag{5.6}$$

## 5.2 Integrals estimation

Estimates the following integrals:

1. $\int_0^{2\pi} e^{x|sin(t)|}tdt$;
2. $\int_1^{10} e^{ix(t^3-4t)}log(t)dt$
3. $\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} e^{ixcos(t)}\sqrt{t}dt$
4. $\int_K e^{-tcos(x)y}\frac{1}{x+y}dxdy \quad$ with $K = [0,\pi] \times [-1,1]$

Let us recall that

$$\int_a^b e^{xf(t)}g(t)dt \sim \sum_{j=1}^m \sqrt{\frac{-2\pi}{xf''(t_j)}}g(t_j)e^{xf(t_j)} \tag{5.7}$$

where $t_j$ are the maximizer of the function $f(t)$ (so $f'(t_j) = 0$ and $f''(t_j) < 0$) and

$$\int_a^b e^{ix\phi(t)}g(t)dt \sim \sum_{j=1}^m \sqrt{\frac{-2\pi}{x|\phi''(t_j)|}}g(t_j)e^{ix\phi(t_j)+sgn(\phi''(t_j))i\frac{\pi}{4}} \tag{5.8}$$

where $t_j$ are the stationary points of the function $\phi(t)$ (so $f'(t_j) = 0$). This formulae hold also in higher dimensions using the obvious multidimensional gaussian integral.

The first case is a Laplace type integral and the function $|sin(x)|$ in the range $[0, 2\pi]$ has a maximum in $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ therefore using Laplace method/approximation decomposing the integration interval into two subintervals (for example $[0, \pi]$ and $[\pi, 2\pi]$) we have

$$\int_0^{\pi} e^{xsin(t)}tdt \sim \sqrt{\frac{-2\pi}{x((sin(x))''|_{x=\frac{\pi}{2}})}}\frac{\pi}{2}e^{xsin(\frac{\pi}{2})} = \sqrt{\frac{2\pi}{x}}\frac{\pi}{2}e^x \tag{5.9}$$

and

$$\int_{\pi}^{2\pi} e^{-xsin(t)}tdt \sim \sqrt{\frac{-2\pi}{x((-sin(x))''|_{x=\frac{3\pi}{2}})}}\frac{3\pi}{2}e^{-xsin(\frac{3\pi}{2})} = \sqrt{\frac{2\pi}{x}}\frac{3\pi}{2}e^x. \tag{5.10}$$

Therefore

$$\int_0^{2\pi} e^{x|sin(t)|}tdt = \sqrt{\frac{2\pi}{x}}e^x + \sqrt{\frac{2\pi}{x}}\frac{3\pi}{2}e^x = \sqrt{\frac{\pi^3}{2x}}[e^x + 3e^x]. \tag{5.11}$$

The second and the third integrals are of Fourier type with

$$\phi_1(t) = t^3 - 4t \Rightarrow \phi_1'(t) = 3t^2 - 4 = 0 \Rightarrow t_0 = \frac{2}{\sqrt{3}} \tag{5.12}$$

and

$$\phi_2(t) = cos(t) \Rightarrow \phi_2'(t) = -sin(t) = 0 \Rightarrow t_k = k\pi \quad with\ k \in \mathbb{Z}. \tag{5.13}$$

Therefore the integrals can be estimated using stationary phase method:

$$\begin{aligned}\int_1^{10} e^{ix(t^3-4t)} log(t)dt \sim \sqrt{\frac{2\pi}{x|\phi_1''(t_0)|}} log(t_0) e^{ix\phi_1(t_0)+sgn(\phi_1''(t_0))\frac{\pi}{4}i} = \\ = \sqrt{\frac{2\pi\sqrt{3}}{12x}} log\left(\frac{2}{\sqrt{3}}\right) e^{i(-\frac{16}{3\sqrt{3}}x+\frac{\pi}{4})},\end{aligned} \tag{5.14}$$

and (note that only $t_0 = \pi$ is in the integration interval)

$$\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} e^{ixcos(t)}\sqrt{t}dt \sim \sqrt{\frac{2\pi}{x|\phi_2''(t_0)|}}\sqrt{t_0} e^{ix\phi_2(t_0)+sgn(\phi_2''(t_0))\frac{\pi}{4}i} = \sqrt{\frac{2\pi}{x}}\sqrt{\pi} e^{i(-x+\frac{\pi}{4})} \tag{5.15}$$

The fourth case is again a Laplace type integral; function $f(x, y) = -ycos(x)$ has gradient given by

$$\nabla f(x, y) = (ysin(x), -cos(x)) \tag{5.16}$$

so

$$\nabla f(x, y) = (ysin(x), -cos(x)) = 0 \Rightarrow y = 0 \cup x = k\frac{\pi}{2} \tag{5.17}$$

and the Hessian is

$$Hf(x, y) = \begin{bmatrix} ycos(x) & sin(x) \\ sin(x) & 0 \end{bmatrix} \Rightarrow det(Hf(x, y)) = -sin^2(x) < 0 \quad for\ x = \frac{\pi}{2}. \tag{5.18}$$

Therefore we have

$$\int_0^{\pi}\int_{-1}^{1} e^{-tcos(x)y}\frac{1}{x+y}dydx \sim \frac{2\pi}{t}\frac{2}{\pi}\frac{1}{\sqrt{1}} = \frac{4}{t} \tag{5.19}$$

# 6 Distributions

## 6.1 Derivatives of distributions

Find the general rule for the derivatives (in the sense of the distributions) of the following functions and compute it on the indicated test function

- $(e^{-x^2})'$ for $\phi(x) = 3\chi_{[-1,1]}(x)$ and $\phi(x) = 2e^{-x}\chi_{[0,+\infty)}$;
- $|t-2|''$ for $\phi(t) = h(t) = \begin{cases} t^3 & if\ t \in [-2, 2]; \\ 0 & otherwise \end{cases}$;

Let us recall that given a function $f(t)$ we can associate (thanks to Riesz representation theorem) the object

$$f(\phi) = \int_{-\infty}^{+\infty} f(t)\phi(t)dt \tag{6.1}$$

where $\phi(t)$ is a test function namely, these are functions in a functional space whose dual is the space of distribution for example in $C_c^\infty$ or in $\mathcal{S}$, this object is a linear functional and it is our distribution. The derivatives in the sense of distribution can be computed as

$$f^{(n)}(\phi) = \int_{-\infty}^{+\infty} f^{(n)}(t)\phi(t)dt = (-1)^n \int_{-\infty}^{+\infty} \phi^{(n)}(t)f(t)dt = (-1)^n f(\phi^{(n)}), \tag{6.2}$$

$\forall \phi(t) \in C_c^\infty, \mathcal{S}$ and where in the second passage we integrate by part $n$ times.
So for the first case we have

$$\begin{aligned}(e^{-x^2})'(\phi) &= -\,(e^{-x^2})(\phi') = -\int_{-\infty}^{+\infty} e^{-x^2}\phi'(x)dx = \\ &= -\,\underbrace{e^{-x^2}\phi(x)\Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} 2xe^{-x^2}\phi(x)dx;\end{aligned} \tag{6.3}$$

so we have

$$(e^{-x^2})'(3\chi_{[-1,1]}) = -3\int_{-1}^{1} 2xe^{-x^2}dx = 0 \tag{6.4}$$

and

$$\begin{aligned}(e^{-x^2})'(2e^{-x}\chi_{[0,+\infty)}) &= -2\int_0^{+\infty} 2xe^{-x^2-x}dx = 4\int_0^{+\infty} \frac{\partial}{\partial t}\left(e^{-x^2-tx}\right)\Big|_{t=1} = \\ &= 4\frac{\partial}{\partial t}\left(\int_0^{+\infty} e^{-x^2-tx}\right)\Big|_{t=1} = 4\frac{\partial}{\partial t}\left(\frac{\sqrt{\pi}}{2}e^{\frac{t^2}{4}}Erfc\left(\frac{t}{2}\right)\right)\Big|_{t=1} = \\ &= \left(\sqrt{\pi}e^{\frac{t^2}{4}}tErfc\left(\frac{t}{2}\right) - 2\right)\Big|_{t=1} = \sqrt{\pi}e^{\frac{1}{4}}Rfrc\left(\frac{1}{2}\right) - 2.\end{aligned} \tag{6.5}$$

For the second case, by definition we have

$$\begin{aligned}(|t-2|)''(\phi) =& (|t-2|)(\phi'') = \int_{-\infty}^{+\infty} |t-2|\phi''(t)dt = \\ &= \int_{-\infty}^{2} (2-t)\phi''(t) + \int_2^{+\infty} (t-2)\phi''(t) = \\ &= \underbrace{(2-t)\phi'(t)\Big|_{-\infty}^{2}}_{=0} + \int_{-\infty}^{2} \phi'(t)dt + \underbrace{(t-2)\phi'(t)\Big|_{2}^{+\infty}}_{=0} - \int_2^{+\infty} \phi'(t)dt\end{aligned} \tag{6.6}$$

where we integrate by parts using in the first integral $f = 2 - t$, $g' = \phi''$ and in the second integral $f = t - 2$, $g' = \phi''$. So we have

$$(|t-2|)''(h) = \int_{-2}^{2} 3t^2 dt = t^3\Big|_{-2}^{+2} = 16 \tag{6.7}$$

## 6.2 The Dirac delta

Compute the following integrals:

- $I = \int_{-\infty}^{+\infty} \big(3\delta(x-1) + 2\delta(x)\big)e^{-x^2+3(x-2)}dx;$
- $II = \int_{-\infty}^{+\infty} 2\delta\big(\frac{x-3}{2}\big)x^3 dx;$
- $III = \int_{-4}^{4}(t-2)^2[\delta'\big(-\frac{t}{3}+\frac{1}{2}\big) + \delta(x-8)]dt;$
- $IV = \int_{-\infty}^{0} x\delta(x^3+1)dx;$

Let us start with some recalling; the fundamental property of the Dirac delta distribution is

$$\int_{-a}^{a} f(x)\delta(x-x_0)dx = \begin{cases} f(x_0) & if\ x_0 \in [-a,a]; \\ 0 & if\ x_0 \notin [-a,a]; \end{cases} \tag{6.8}$$

this is due to the very definition of distribution as linear functional from a space to its dual. Moreover from the definition of distributional derivative follows that

$$\begin{aligned} \int_{-a}^{a} f(x)\delta^{(n)}(x-x_0)dx &= (-1)^n \int_{-a}^{a} f^{(n)}(x)\delta(x-x_0)dx = \\ &= \begin{cases} (-1)^n f^{(n)}(x_0) & if\ x_0 \in [-a,a]; \\ 0 & if\ x_0 \notin [-a,a]. \end{cases} \end{aligned} \tag{6.9}$$

Other important and useful properties are (expressed with abuse of language since these properties are true only when we think the delta as a linear functional):

- $\delta(ax) = \frac{\delta(x)}{|a|};$
- $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}.$

where $x_i$ are the zeroes of the function $f(x)$. Now we can start. In case $I$ we have

$$\int_{-\infty}^{+\infty} (3\delta(x-1)+2\delta(x))e^{-x^2+3(x-2)}dx = 3e^{-1^2+3(1-2)} + 2e^{-0^2+3(0-2)} = \frac{3}{e^4} + \frac{2}{e^6}. \tag{6.10}$$

Case $II$ can be solved using rescaling property of delta distribution, indeed

$$\int_{-\infty}^{+\infty} 2\delta\left(\frac{x-3}{2}\right)x^3 dx = 2\int_{-\infty}^{+\infty} 2\delta(x-3)x^3 dx = 43^3 = 108. \tag{6.11}$$

Case $III$ is a little more complex; first of all note that since $x = 8 \notin [-4,4]$ the second piece does not play any role. Now using the definition of distributional derivatives we get

$$\int_{-4}^{4} (t-2)^2 \delta'\Big(-\frac{t}{3}+\frac{1}{2}\Big)dt = -\int_{-4}^{4}\Big[(t-2)^2\Big]'\delta\Big(-\frac{t}{3}+\frac{1}{2}\Big)dt, \tag{6.12}$$

and using composition rule for delta distribution we can rewrite it as

$$\delta\Big(-\frac{t}{3}+\frac{1}{2}\Big) = \frac{\delta\big(t-\frac{3}{2}\big)}{|-\frac{1}{3}|} = 3\delta\Big(t-\frac{3}{2}\Big) \tag{6.13}$$

since

$$-\frac{t}{3}+\frac{1}{2} = 0 \Rightarrow t = \frac{3}{2}, \quad \Big(-\frac{t}{3}+\frac{1}{2}\Big)' = -\frac{1}{3}. \tag{6.14}$$

In the end

$$-\int_{-4}^{4}\Big[(t-2)^2\Big]'\delta\Big(-\frac{t}{3}+\frac{1}{2}\Big)dt = -3\int_{-4}^{4}\Big[(t-2)^2\Big]'\delta\Big(t-\frac{3}{2}\Big)dt = -3\Big[(t-2)^2\Big]'\Big|_{t=\frac{3}{2}} = 3. \tag{6.15}$$

Case $IV$ is similar, indeed

$$x^3+1 = 0 \Rightarrow x = -1, \quad \Big(x^3+1\Big)' = 3x^2; \tag{6.16}$$

therefore

$$\int_{-\infty}^{0} x\delta(x^3+1)dx = \int_{-\infty}^{0} x\frac{\delta(x+1)}{|3|} = -\frac{1}{3}. \tag{6.17}$$

## 6.3 ODEs and weak solutions

Solve the following Cauchy problems:

1. $xy''(x) - 2y'(x) + (x+\frac{2}{x})y(x) = \delta(x-\pi), \quad y(\frac{\pi}{2}) = y(\frac{3\pi}{2}) = 0$
2. $xy'(x) + y(x) = \delta(x-2) + \Theta(x-3), \quad y(1) = 1;$
3. $y'''(x) + \frac{y''(x)}{x} - 2\frac{y'(x)}{x^2} + 2\frac{y(x)}{x^3} = 2\delta(x-2), \quad y(1) = 1, y'(1) = y''(1) = 0.$

In general the solution of a ODE is given as $y(x) = y_0(x) + y_p(x)$, where $y_0(x)$ is the solution of the homogeneous equation while $y_p(x)$ is a particular solution of our problem. Let us start with case 1. Let us massage the homogeneous equation

$$\begin{aligned}
&y_0'' - \frac{2}{x}y_0' + \frac{x^2+2}{x^2}y_0 = 0 \quad \underbrace{\Rightarrow}_{y_0(x)=e^{\int \frac{1}{x}dx}g(x)=xg(x)} \quad (xg)'' - \frac{2}{x}(xg)' + \frac{x^2+2}{x^2}xg = 0 \Rightarrow \\
&\Rightarrow g' + g' + xg'' - \frac{2}{x}(g+xg') + \frac{x^2+2}{x}g = 0 \Rightarrow xg'' + \Big(-\frac{2}{x}+\frac{x^2+2}{x}\Big)g = 0 \Rightarrow \\
&\Rightarrow g'' + g = 0;
\end{aligned} \tag{6.18}$$

this is a simple harmonic oscillator and the solutions are sine and cosine. Therefore $g(x) = acos(x) + bsin(x)$ and

$$y_0(x) = axcos(x) + bxsin(x). \tag{6.19}$$

When $x < \pi$ the equation is homogeneous and the solution is $y_0(x)$, while for $x > \pi$ something happens: since the second derivative has a Dirac delta behavior, we expect that the first derivative has a Heaviside behavior and the function to be continuous in the point $\pi$ (obviously this is true in the sense of distribution and therefore we are looking a weak or distributional solution). So we can write a modified solution after $x = \pi$ changing the coefficients but with the same functional form since for $x > \pi$ the equation is still homogeneous. Let us write the general solution before and after $x = \pi$ as

$$\begin{cases} y_{x<\pi}(x) = a_1xcos(x) + b_1xsin(x) & if\ x < \pi, \\ y_{x>\pi}(x) = a_2xcos(x) + b_2xsin(x) & if\ x > \pi \end{cases}; \tag{6.20}$$

and we need to impose continuity of the function and step singularity of $\frac{1}{a(x_0)} = \frac{1}{x_0} = \frac{1}{\pi}$ of the first derivative (where $a(x)$ is the coefficient of the highest derivative order term of the equation). We have

$$y_{x<\pi}(\pi^-) = y_{x>\pi}(\pi^+) \Rightarrow a_1 = a_2 \tag{6.21}$$

and

$$y'_{x>\pi}(\pi^+) - y'_{x<\pi}(\pi^-) = \frac{1}{\pi} \Rightarrow -a_2 - b_2\pi + a_1 + b_1\pi = \frac{1}{\pi} \Rightarrow b_1 - b_2 = \frac{1}{\pi^2}. \tag{6.22}$$

The general solution is therefore

$$\begin{cases} y_{x<\pi}(x) = a_1xcos(x) + b_1xsin(x) & if\ x < \pi, \\ y_{x>\pi}(x) = a_1xcos(x) + \left(b_1 - \frac{1}{\pi^2}\right)xsin(x) & if\ x > \pi \end{cases} \tag{6.23}$$

and imposing the boundary conditions we get

$$y\left(\frac{\pi}{2}\right) = b_1\frac{\pi}{2} = 0, \quad y\left(\frac{3\pi}{2}\right) = -b_1\frac{3\pi}{2} - \left(b_1 - \frac{1}{\pi^2}\right)\frac{3\pi}{2} = 0 \tag{6.24}$$

but this system has no solution, therefore our Cauchy problem does not admit any solution. Let us consider case 2, this is an Euler equation and the homogeneous solution can be find in the form $y_0(x) = c_1x^\alpha$:

$$x\alpha x^{\alpha-1} + x^\alpha = 0 \Rightarrow \alpha + 1 = 0 \tag{6.25}$$

and so $\alpha = -1$ and $y_0(x) = \frac{c_1}{x}$. The Heaviside term enter in the game only when $x > 3$ and here the function must be continuous, while the Dirac term makes the function discontinuous in $x=2$. Let us consider first the Dirac term, the solution is modified but the equation is still homogeneous; we write

$$\begin{cases} y_{x<2}(x) = \frac{c_1}{x} & if\ x<2, \\ y_{x>2}(x) = \frac{c_2}{x} & if\ x>2 \end{cases}; \tag{6.26}$$

and we impose the discontinuity at $x=2$

$$y_{x>2}(2^+) - y_{x<2}(2^-) = \frac{1}{2} \Rightarrow \frac{c_2}{2} - \frac{c_1}{2} = \frac{1}{2} \Rightarrow c_2 = c_1 + 1. \tag{6.27}$$

Now the Heaviside term; for $x>3$ the equation becomes

$$xy'(x) + y(x) = 1, \tag{6.28}$$

the homogeneous solution is the same as before but now we have also a particular solution. The simpler choice is $y_p(x) = 1$ and the complete solution is $y_{x>3}(x) = \frac{c_3}{x} + 1$ (while $y_{x<3}(x) = y_{x>2}(x)$) where $c_3$ has to be find imposing continuity of the function in $x=3$,

$$y_{x<3}(3^-) = y_{x>3}(3^+) \Rightarrow \frac{c_1+1}{3} = \frac{c_3}{3} + 1 \Rightarrow c_3 = c_1 - 2. \tag{6.29}$$

The general solution is therefore

$$y(x) = \begin{cases} \frac{c_1}{x} & if\ x<2, \\ \frac{c_1}{x} & if\ 2<x<3, \\ \frac{c_1-3}{x} + 1 & if\ x>3 \end{cases}; \tag{6.30}$$

and imposing the initial condition we find

$$y(1) = c_1 = 1; \tag{6.31}$$

so

$$y(x) = \begin{cases} \frac{1}{x} & if\ x<2, \\ \frac{2}{x} & if\ 2<x<3, \\ -\frac{1}{x} + 1 & if\ x>3 \end{cases}; \tag{6.32}$$

Case 3 is again an Euler equation, indeed, in the homogeneous case for $x \neq 2$, we have (searching a solution of the form $y(x) = x^\alpha$)

$$\begin{aligned} x^3y''' + x^2y'' - 2xy' + 2 = 0 &\Rightarrow \alpha(\alpha-1)(\alpha-2) + \alpha(\alpha-1) - 2\alpha + 2 = 0 \\ &\Rightarrow (\alpha-1)[\alpha(\alpha-2) + \alpha - 2] = 0 \\ &\Rightarrow (\alpha-1)(\alpha-2)[\alpha+1] = 0 \Rightarrow \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = -1. \end{aligned} \tag{6.33}$$

So the solution is

$$y(x) = a_1x + b_1x^2 + \frac{c_1}{x}. \tag{6.34}$$

Since there is a Dirac term, we expect a step discontinuity in the second derivative and continuity in the function and it first derivative in $x = 2$; let us write the solution for $x > 2$, here the equation is still homogeneous but we need to change the coefficents. In the end we have

$$\begin{cases} y_{x<2}(x) = a_1 x + b_1 x^2 + \frac{c_1}{x} & if\ x < 2, \\ y_{x>2}(x) = a_2 x + b_2 x^2 + \frac{c_2}{x} & if\ x > 2 \end{cases}, \tag{6.35}$$

and imposing continuity and discontinuity we get

$$\begin{cases} y_{x<2}(2^-) = y_{x>2}(2^+) = 2a_1 + 4b_1 + \frac{c_1}{2} = 2a_2 + 4b_2 + \frac{c_2}{2}, \\ y'_{x<2}(2^-) = y'_{x>2}(2^+) = a_1 + 4b_1 - \frac{c_1}{4} = a_2 + 4b_2 - \frac{c_2}{4}, \\ y''_{x>2}(2^+) - y''_{x<2}(2^-) = 2b_2 + 2\frac{c_2}{8} - 2b_1 - 2\frac{c_1}{8} = 2 \end{cases} ; \tag{6.36}$$

instead of solve this linear system in general, we solve it using the boundary conditions

$$\begin{aligned} y_{x<2}(1) = 1 &\Rightarrow a_1 + b_1 + c_1 = 1, \\ y'_{x<2}(1) = 0 &\Rightarrow a_1 + 2b_1 - c_1 = 0, \\ y''_{x<2}(1) = 0 &\Rightarrow +2b_1 + 2c_1 = 0, \end{aligned} \tag{6.37}$$

whose solution is given by $a_1 = 1, b_1 = -\frac{1}{3}, c_1 = \frac{1}{3}$. Therefore we have

$$\begin{cases} 2 - \frac{4}{3} + \frac{1}{6} = 2a_2 + 4b_2 + \frac{c_2}{2}, \\ 1 - \frac{4}{3} - \frac{1}{12} = a_2 + 4b_2 - \frac{c_2}{4}, \\ \frac{2}{3} - \frac{1}{12} - 2b_2 - 2\frac{c_2}{8} = 2 \end{cases} ; \tag{6.38}$$

and so

$$a_2 = \frac{11}{2}, \quad a_2 = \frac{11}{6}, \quad c_2 = 9. \tag{6.39}$$

In the end the solution is

$$y(x) = \begin{cases} y_{x<2}(x) = x + -\frac{x^2}{3} + \frac{1}{3x} & if\ x < 2, \\ y_{x>2}(x) = \frac{11x}{2} + \frac{11x^2}{6} + \frac{9}{x} & if\ x > 2 \end{cases}. \tag{6.40}$$

# 7 Finite dimensional linear spaces and euclidean spaces

## 7.1 Matrices and invariant quantities

Given the following linear application compute the representative matrices and the invariant quantities:

- $g(x, y) = (5x - 10y, x - 2y) : \mathbb{R}^2 \to \mathbb{R}^2$;

- $h(x,y,z) = (-10y, x-y, x+y+z) : \mathbb{R}^3 \to \mathbb{R}^3$;

To understand from which matrix function $f(x,y)$ is represented we chose a basis and than we can compute the trace, the determinant and the characteristic polynomial. For the firs case, let us choose the standard $\mathbb{R}^2$ basis, we have

$$\begin{aligned} g(1,0) &= (5,1), \\ g(0,1) &= (-10,-2). \end{aligned} \tag{7.1}$$

Therefore we have the following matrix representation

$$A_{g(x,y)} = \begin{bmatrix} 5 & -10 \\ 1 & -2 \end{bmatrix}; \tag{7.2}$$

the invariants are

$$\begin{aligned} &ch_{A_{g(x,y)}} = Det(A_{g(x,y)} - \lambda\mathcal{I}) = (5-\lambda)(-2-\lambda) + 10 = \lambda^2 - 3\lambda; \\ &Tr(A_{g(x,y)}) = 3; \\ &Det(A_{g(x,y)}) = -10 + 10 = 0. \end{aligned} \tag{7.3}$$

In the second case we choose the standard basis of $\mathbb{R}^3$, we have

$$\begin{aligned} h(1,0,0) &= (0,1,1), \\ h(0,1,0) &= (-10,-1,1), \\ h(0,0,1) &= (0,0,1). \end{aligned} \tag{7.4}$$

Therefore we have the following matrix representation

$$A_{h(x,y,z)} = \begin{bmatrix} 0 & -10 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \tag{7.5}$$

andnthe invariants are

$$\begin{aligned} &ch_{A_{h(x,y,z)}} = Det(A_{h(x,y,z)} - \lambda\mathcal{I}) = (1-\lambda)[(-\lambda)(-1-\lambda) + 10] = -\lambda^3 - 9\lambda + 10; \\ &Tr(A_{h(x,y,z)}) = 0; \\ &Det(A_{h(x,y,z)}) = 10; \\ &I_2 = \frac{1}{2}[(Tr(A_{h(x,y,z)}))^2 - Tr(A^2_{h(x,y,z)})] = 9. \end{aligned} \tag{7.6}$$

## 7.2 Invariant subspaces

Determine the invariant subspaces of the following matrix

- $h(z_1, z_2) = (z_1 + z_2, z_1 - z_2) : \mathbb{C}^2 \to \mathbb{C}^2$

We can think $\mathbb{C}^2$ as $\mathbb{R}^4$; therefore we can write

$$h(x_1, y_1, x_2, y_2) = (x_1 + x_2, y_1 + y_2, x_1 - x_2, y_1 - y_2) : \mathbb{R}^4 \to \mathbb{R}^4; \tag{7.7}$$

choosing the standard basis of $\mathbb{R}^4$ we have

$$\begin{aligned} h(1,0,0,0) &= (1,0,1,0), \\ h(0,1,0,0) &= (0,1,0,1), \\ h(0,0,1,0) &= (1,0,-1,0), \\ h(0,0,0,1) &= (0,1,0,-1), \end{aligned} \tag{7.8}$$

therefore the representing matrix is

$$A_{h(x_1,y_1,x_2,y_2)} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}. \tag{7.9}$$

For the invariant subspaces, note that every vector of the form $(x_1, 0, x_2, 0)$ is mapped into $(x_1 + x_2, 0, x_1 - x_2, 0)$ and every vector of the form $(0, y_1, 0, y_2)$ is mapped into $(0, y_1 + y_2, 0, y_1 - y_2)$. The first is the $\mathbb{R}^2$ subspace of $\mathbb{C}^2$ while the second is $i\mathbb{R}^2$ (with abuse of language) subspace of $\mathbb{C}^2$

## 7.3 Nullity + Rank

Given the following linear application say if they are injective, surjective or bijective

- $h(x, y, z) = (2x, x - 2y, 2y - z) : \mathbb{R}^3 \to \mathbb{R}^3$
- $g(x, y, z) = (2x + z, x - 2y) : \mathbb{R}^3 \to \mathbb{R}^2$

If we have a liner map $f$ from $V$ to $W$ (both on the fild $K$), it is injective if $dim_K(Ker(f)) = 0$ and it is surjective if $dim_K(Image(f)) = dim_K(W)$; morover the following is a fundamental result

$$dim_K(Image(f)) + dim_K(Ker(f)) = n = Rank(A) + Null(A) \tag{7.10}$$

where $A$ is the representing matrix of $f$ and $n = dim_K(V)$. If the linear map is both injective and surjective than it is bijective. We have the following implications:

- if $dim_K(W) = dim_K(V)$ than $f$ is injective if and only if it is surjective (this is obvious: if the image of the application has the same dimension of the starting and arriving vector spaces, the kernel must be trivial);
- if $dim_K(V) > dim_K(W)$ than $f$ is not injective (this is obvious again: if the dimension of the starting vector space is grater than the one of the arriving vector space, some subspace of the starting vector space must be contained in the kernel of the application);

- if $dim_K(W) > dim_K(V)$ than $f$ is not surjective.

In the first case we have $dim_K(W) = dim_K(V) = 3$ so we only need to show that $f$ is surjective. Let us construct the representing matrix choosing the standard basis of $\mathbb{R}^3$:

$$h(1,0,0) = (2,1,0), \quad h(0,1,0) = (0,-2,2), \quad h(0,0,1) = (0,0,-1), \tag{7.11}$$

therefore the representing matrix is

$$A_{h(x,y,z)} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & -1 \end{bmatrix}. \tag{7.12}$$

Its determinant is not vanishing and therefore its rank is maximal: the application is bijective. In the second application we have $dim_K(V) > dim_K(W)$ and so it is not injective but is can be surjective: let us see. The representing matrix, choosing the standard basis of $\mathbb{R}^3$ is

$$A_{g(x,y,z)} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix}; \tag{7.13}$$

the two rows are independent and so the rank is $Rank = 2 = dim_K(W)$: the function is surjective.

## 7.4 Euclidean spaces

Given the following spaces endowed by the prescribed scalar product say if they are Euclidean spaces and orthonormalize the reported vectors if the space is Euclidean.

- the space of matrix $Mat(\mathbb{C}, 2)$ with the scalar product $(X,Y) = Tr(X^\dagger Y)$, $A = \begin{bmatrix} i & 2 \\ i & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2i \\ 1 & -1 \end{bmatrix}$;
- the space of matrix $Mat(\mathbb{R}, 2)$ with the scalar product $(X,Y) = Tr(XY^2X^T)$, $A = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$;
- the space of polynomial of degree 2 in $[0,1]$ with the scalar product $(p(z), q(z) = \int_0^1 \overline{p(z)} q(z) dz$; $p(z) = 1 + 3iz^2$, $q(z) = 3 + 2z - z^2$
- the space of polynomial of degree 2 in $[0,1]$ with the scalar product $(p(z), q(z)) = \sum_{n=0}^{2} \bar{p}_n q_n$ with $p_n$ the coefficients of the $n$ degree monomial of $p(z)$ and similar for $q_n$, $p(z) = 1 + 3iz^2$, $q(z) = 3 + 2z - z^2$

The first thing to do is to show that they are scalar products, namely that for every $x, y \in V$ we have

1. $(x, y) = \overline{(y, x)}$;
2. $(x, \lambda y) = \lambda(x, y) \quad \text{for} \quad \lambda \in \mathbb{C}$;
3. $(x, y + z) = (x, y) + (x, z)$;
4. $(x, x) \geq 0 \;\text{ and }\; (x, x) = 0 \;\text{ iff }\; x = 0$.

The first one is a true scalar product due to the properties of the trace indeed:

1. $Tr(X^\dagger Y) = Tr(Y^\dagger X)$;
2. $Tr(X^\dagger \lambda Y) = \lambda Tr(X^\dagger Y)$;
3. $Tr(X^\dagger (Y + Z)) = Tr(X^\dagger Y + X^\dagger Z) = Tr(X^\dagger Y) + Tr(X^\dagger Z)$;
4. $Tr(X^\dagger X) = \sum_{i,j=1}^{n} \bar{x}_{ij} x_{ij} \geq 0$;

so $Mat(\mathbb{C}, 2)$ with $Tr(X^\dagger Y)$ as a product is an euclidean space. The second one is not a scalar product since it does not respect, for example, point 2; indeed $(X, \lambda Y) = \lambda^2 (X, Y)$. Therefore $Mat(\mathbb{R}, 2)$ with $Tr(XY^2X^\dagger)$ is not an euclidean space. The third case is a scalar product indeed

1. $\overline{\int_0^1 \overline{q(z)} p(z) dz} = \int_0^1 \overline{p(z)} q(z) dz$;
2. $\int_0^1 \overline{p(z)} \lambda q(z) dz = \lambda \int_0^1 \overline{p(z)} q(z) dz$;
3. $\int_0^1 \overline{p(z)} (q(z) + l(z)) dz = \int_0^1 \overline{p(z)} q(z) dz + \int_0^1 \overline{p(z)} l(z) dz$;
4. $\int_0^1 \overline{p(z)} p(z) dz \geq 0$

the last case is identical. The space of polynomial with these two product is, in both cases, an euclidean space.
Now we have to orthonomalize the reported vectors; this is done using Gram-Schmidt procedure

$$e^{(1)} = \frac{x^{(1)}}{||x^{(1)}||}, \quad e^{(j)} = \frac{x^{(j)} - \sum_{k=1}^{j-1} (e^{(k)}, x^{(j)}) e^{(k)}}{||x^{(j)} - \sum_{k=1}^{j-1} (e^{(k)}, x^{(j)}) e^{(k)}||} \quad 2 \leq j \leq n. \tag{7.14}$$

In the fist case we have, since $||A|| = \sqrt{Tr(A^\dagger A)}$,

$$A^\dagger A = \begin{bmatrix} -i & -i \\ 2 & -2 \end{bmatrix} \begin{bmatrix} i & 2 \\ i & -2 \end{bmatrix} = \begin{bmatrix} 1+1 & -2i+2i \\ 2i-2i & 4+4 \end{bmatrix} \Rightarrow ||A|| = \sqrt{10}; \tag{7.15}$$

therefore

$$e^{(1)} = \frac{A}{\sqrt{10}} \tag{7.16}$$

while

$$e^{(2)} = \frac{B - Tr\left(\frac{A^\dagger}{\sqrt{10}}B\right)B}{||B - Tr\left(\frac{A^\dagger}{\sqrt{10}}B\right)B||} \tag{7.17}$$

where

$$\begin{aligned} Tr\left(\frac{A^\dagger}{\sqrt{10}}B\right)B &= \frac{1}{\sqrt{10}}Tr\left(\begin{bmatrix}-i & -i\\ 2 & -2\end{bmatrix}\begin{bmatrix}0 & 2i\\ 1 & -1\end{bmatrix}\right)B = \frac{1}{\sqrt{10}}Tr\left(\begin{bmatrix}-i & 2+i\\ -2 & 4i+2\end{bmatrix}\right)B \\ &= \frac{2+3i}{\sqrt{10}}B \end{aligned} \tag{7.18}$$

and

$$\begin{aligned} B - \frac{2+3i}{\sqrt{10}}B = \begin{bmatrix}0 & 2i\\ 1 & -1\end{bmatrix} - \frac{2+3i}{\sqrt{10}}\begin{bmatrix}0 & 2i\\ 1 & -1\end{bmatrix} = \begin{bmatrix}0 & 2i\\ 1 & -1\end{bmatrix} = \\ = \frac{1}{\sqrt{10}}\begin{bmatrix}0 & 6+i(2\sqrt{10}-4)\\ \sqrt{10}-2-3i & 2-\sqrt{10}+3i\end{bmatrix}; \end{aligned} \tag{7.19}$$

therefore

$$\begin{aligned} &||B - \frac{2+3i}{\sqrt{10}}B|| = \\ &= \frac{1}{\sqrt{10}}\sqrt{Tr\left(\begin{bmatrix}0 & \sqrt{10}-2+3i\\ 6-i(2\sqrt{10}-4) & 2-\sqrt{10}+3i\end{bmatrix}\begin{bmatrix}0 & 6+i(2\sqrt{10}-4)\\ \sqrt{10}-2-3i & 2-\sqrt{10}+3i\end{bmatrix}\right)} = \\ &= \sqrt{\frac{-24\sqrt{10}+138}{10}}. \end{aligned} \tag{7.20}$$

In the end

$$e^{(1)} = \frac{1}{\sqrt{10}}\begin{bmatrix}i & 2\\ i & -2\end{bmatrix}, \quad e^{(2)} = \frac{\frac{1}{\sqrt{10}}}{\sqrt{\frac{-24\sqrt{10}+138}{10}}}\begin{bmatrix}0 & 6+i(2\sqrt{10}-4)\\ \sqrt{10}-2-3i & 2-\sqrt{10}+3i\end{bmatrix} \tag{7.21}$$

In the third case we have

$$||p(z)|| = \sqrt{\int_0^1 (1-3iz^2)(1+3iz^2)dz} = \sqrt{\int_0^1 (1+9z^4)dz} = \sqrt{\left(z+\frac{9}{5}z^5\right)\Big|_0^1} = \sqrt{\frac{14}{5}}, \tag{7.22}$$

so

$$e^{(1)} = \frac{p(z)}{\sqrt{\frac{14}{5}}} \tag{7.23}$$

The second basis vector is

$$e^{(2)} = \frac{q(z) - \int_0^1\left(\frac{\overline{p(z)}}{\sqrt{\frac{14}{5}}}q(z)dz\right)q(z)}{||q(z) - \int_0^1\left(\frac{\overline{p(z)}}{\sqrt{\frac{14}{5}}}q(z)dz\right)q(z)||} \tag{7.24}$$

where

$$\int_0^1 \left(\frac{\overline{p(z)}}{\sqrt{\frac{14}{5}}}q(z)\right)q(z) = \sqrt{\frac{5}{14}}q(z)\int_0^1 (1-3iz^2)(3+2z-z^2)dz = \sqrt{\frac{5}{14}}\left(\frac{11}{3}-\frac{39}{10}i\right)q(z) \tag{7.25}$$

and

$$q(z)-\sqrt{\frac{5}{14}}\left(\frac{11}{3}-\frac{39}{10}i\right)q(z) =$$
$$= 3\left(1-\sqrt{\frac{5}{14}}\left(\frac{11}{3}-\frac{39}{10}i\right)\right)+2\left(1-\sqrt{\frac{5}{14}}\left(\frac{11}{3}-\frac{39}{10}i\right)\right)z-\left(1-\sqrt{\frac{5}{14}}\left(\frac{11}{3}-\frac{39}{10}i\right)\right)z^2; \tag{7.26}$$

therefore

$$||q(z)-\sqrt{\frac{5}{14}}\left(\frac{11}{3}-\frac{39}{10}i\right)q(z)|| = ... \tag{7.27}$$

In the end we have

$$e^{(1)} = \sqrt{\frac{5}{14}}+3\sqrt{\frac{5}{14}}iz^2,$$
$$e^{(2)} = \frac{3\left(1-\sqrt{\frac{5}{14}}\left(\frac{11}{3}-\frac{39}{10}i\right)\right)+2\left(1-\sqrt{\frac{5}{14}}\left(\frac{11}{3}-\frac{39}{10}i\right)\right)z-\left(1-\sqrt{\frac{5}{14}}\left(\frac{11}{3}-\frac{39}{10}i\right)\right)z^2}{...}. \tag{7.28}$$

The last case is simpler:

$$||p(z)|| = \sqrt{1+9} = \sqrt{10} \tag{7.29}$$

so

$$e^{(1)} = \frac{p(z)}{\sqrt{10}}; \tag{7.30}$$

The second basis vector is

$$e^{(2)} = \frac{q(z)-\frac{q(z)}{\sqrt{10}}\sum_{n=0}^2 \overline{p(z)}q(z)}{||q(z)-\frac{q(z)}{\sqrt{10}}\sum_{n=0}^2 \overline{p(z)}q(z)||} \tag{7.31}$$

where

$$q(z)-\frac{q(z)}{\sqrt{10}}\sum_{n=0}^2 \overline{p(z)}q(z) = 3\left(1-\frac{3+3i}{\sqrt{10}}\right)+2\left(1-\frac{3+3i}{\sqrt{10}}\right)z-\left(1-\frac{3+3i}{\sqrt{10}}\right)z^2 \tag{7.32}$$

and

$$||q(z)-\frac{q(z)}{\sqrt{10}}\sum_{n=0}^2 \overline{p(z)}q(z)|| = \frac{126-27\sqrt{10}}{5}+\frac{56-12\sqrt{10}}{5}+\frac{14-3\sqrt{10}}{5} = \frac{196-42\sqrt{10}}{5}; \tag{7.33}$$

in the end

$$
\begin{aligned}
e^{(1)} &= \frac{1}{\sqrt{10}}\left[1+3iz^2\right], \\
e^{(2)} &= \frac{5}{196-42\sqrt{10}}\left[3\left(1-\frac{3+3i}{\sqrt{10}}\right)+2\left(1-\frac{3+3i}{\sqrt{10}}\right)z-\left(1-\frac{3+3i}{\sqrt{10}}\right)z^2\right].
\end{aligned} \tag{7.34}
$$

# 8 Eigenvalue problems and matrix functions

## 8.1 Classification of matrices

Classify the following matrices:

1. $\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & -i & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$;

2. $\begin{bmatrix} 0 & 1+i & 1+i \\ 1-i & 0 & 1+i \\ 1-i & 1-i & 0 \end{bmatrix}$;

3. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$;

4. $\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

The first matrix satisfies the relation

$$
\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & -i & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}^T \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & -i & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{8.1}
$$

therefore it is unitary. The second one is hermitian since

$$
\begin{bmatrix} 0 & 1+i & 1+i \\ 1-i & 0 & 1+i \\ 1-i & 1-i & 0 \end{bmatrix}^\dagger = \begin{bmatrix} 0 & 1+i & 1+i \\ 1-i & 0 & 1+i \\ 1-i & 1-i & 0 \end{bmatrix}. \tag{8.2}
$$

The third matrix is antisymmetric while the last one is symmetric.

## 8.2 Matrices determination

Determine the matrix under the following conditions:

- the generic $3 \times 3$ antisymmetric matrix $A$ satisfying the conditions $A^2 = \mathcal{I}$ and $Tr(A^2) = 0$;
- the complex matrices $B$ with eigenvectors $\vec{v}_1 = (1, i, 0)$, $\vec{v}_2 = (0, 0, 1)$, $\vec{v}_3 = (i, 1, 0)$ such that $det(B) = 1, Tr(B) = 0, A\vec{v}_2 = \vec{v}_2, Im(\lambda_1) > Im(\lambda_2)$;
- determine the matrix $C$ such that $C \in SO(2)$ and $C \in SU(2)$;
- determine the $2 \times 2$ matrix $D$ such that which leaves unchanged the matrix $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

For the first case, the generic $3 \times 3$ antisymmetric matrix is given by

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}, \tag{8.3}$$

its square is

$$A^2 = \begin{bmatrix} -a^2 - b^2 & -bc & ac \\ -bc & -a^2 - c^2 & -ab \\ ac & -ab & -b^2 - c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{8.4}$$

so

$$-bc = 0, \quad ac = 0, \quad -ab = 0, \quad -a^2 - b^2 = 1, \quad -a^2 - c^2 = 1, \quad -b^2 - c^2 = 1; \tag{8.5}$$

and

$$Tr(A^2) = -2a^2 - 2b^2 - 2c^2 = 0. \tag{8.6}$$

This set of equations has no solutions. For the second case we know the matrix is diagonalizable, therefore the condition can be rewritten as

$$\lambda_2 = 1, \quad \lambda_1\lambda_2\lambda_3 = 1, \quad \lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{8.7}$$

The system has solutions

$$\left(\frac{-1 - i\sqrt{3}}{2}, 1, \frac{-1 + i\sqrt{3}}{2}\right), \quad \left(\frac{-1 + i\sqrt{3}}{2}, 1, \frac{-1 - i\sqrt{3}}{2}\right) \tag{8.8}$$

but the requirement $Im(\lambda_1) > Im(\lambda_3)$ fix the solution

$$\left(\frac{-1 + i\sqrt{3}}{2}, 1, \frac{-1 - i\sqrt{3}}{2}\right), \tag{8.9}$$

which is the diagonal form of the matrix $B$. To obtain the general form in the canonical base of $\mathbb{R}^3$, we need to change basis; using the orthonormalized eigenvectors we build up the base change matrix

$$BCM = \begin{bmatrix} 1 & 0 & i \\ i & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{8.10}$$

whose inverse is

$$(BCM)^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} & 0 \\ 0 & 0 & 1 \\ -\frac{i}{2} & \frac{1}{2} & 0 \end{bmatrix}; \tag{8.11}$$

using

$$B = (BCM)B_{diag}(BCM)^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8.12}$$

The third case is simply: we know that

$$\begin{aligned} SO(2) &:= \{M \in GL(2,\mathbb{R}) | det(M) = 1, M^T M = \mathcal{I}\}, \\ SU(2) &:= \{M \in GL(2,\mathbb{C}) | det(M) = 1, M^\dagger M = \mathcal{I}\} \end{aligned} \tag{8.13}$$

Therefore we need to compute the general matrix in $SO(2)$; this is the matrix

$$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{8.14}$$

need to satisfy the conditions

$$a^2 + b^2 = 1, \quad ac = -bd, \quad c^2 + d^2 = 1, \quad ad - bc = 1 \tag{8.15}$$

whose solution is

$$a = \pm d, \quad b = \pm\sqrt{1-a^2}, \quad d = \pm\sqrt{1-d^2}. \tag{8.16}$$

In the last case we need to require

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{8.17}$$

therefore we have

$$-bc + ad = 1, \quad bc - ad = -1 \tag{8.18}$$

so

$$-bc + ad = 1. \tag{8.19}$$

## 8.3 Matrix functions

Given the following matrices compute the corresponding function:

- for a generic $3 \times 3$ matrix $A$ with the following spectral decomposition $A = \sum_{k=1}^{3} \lambda^k P^{(k)}$ with $\lambda^k = e^{i\pi k}$ compute $A^3$;
- given $P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$ compute the function $P^n \quad \forall n \in \mathbb{N}$;
- given the odd function $f(x)$ such that $f(-1 \mp \sqrt{3}) = 2$, compute the eigenvalues of $f(A)$ with $A = (I + \sum_i \sigma_i)$, where $\sigma_i$ are the Pauli matrices;
- given the matrix $A = f(a\mathcal{I} + b\vec{n} \cdot \vec{\sigma})$, where $a, b$ are real positive parameters and $\vec{n} = \frac{1}{\sqrt{2}}(1, 0, 1)$. Determine its general form in the Pauli basis and specialize for $a = 0, b = 1$ and $f(\lambda_i) = 1$ where $\lambda_i$ are the eigenvalues of $a\mathcal{I} + b\vec{n} \cdot \vec{\sigma}$.

In the first case we have

$$\begin{aligned} A^3 &= \left( \sum_{k=1}^{3} \lambda^k P^{(k)} \right) = \sum_{k,j,i=1}^{3} \lambda^k P^{(k)} \lambda^j P^{(j)} \lambda^i P^{(i)} = \sum_{k,j,i=1}^{3} \lambda^k \delta_{kj} \lambda^j \delta_{ji} \lambda^i P^{(i)} = \\ &= \sum_{i=1}^{3} (\lambda^i)^3 P^{(i)} = -P^{(1)} + P^{(2)} - P^{(3)} \end{aligned} \tag{8.20}$$

where we used the idempodence property of the projectors. For the second case we note that

$$P^2 = P, \tag{8.21}$$

therefore

$$P^n = \begin{cases} P & if\ n = 2k; \\ P & if\ n = 2k + 1; \end{cases} \tag{8.22}$$

where $k \in \mathbb{N}$. In the third case we use Cayley-Hamilton theorem: essentially every matrix is a root of its characteristic polynomial. Therefore for a matrix $n \times n$, we can always reduce its power grater than $n - 1$ to a sum of the first $n - 1$ powers; so every matrix function can be rewritten as

$$f(A) = \sum_{m=0}^{n-1} f_m A^m, \tag{8.23}$$

and the coefficients can be find requiring that on the diagonal form of the matrix we have

$$f(\lambda_j) = \sum_{m=0}^{n-1} f_m \lambda_j^m \tag{8.24}$$

where $\lambda_j$ are the eigenvalues. In our case we have

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 0 \end{bmatrix}, \tag{8.25}$$

whose eigenvalues are $\lambda_- = 1 - \sqrt{3}$ and $\lambda_+ = 1 + \sqrt{3}$. Now

$$f(A) = f_0 \mathcal{I} + f_1 A \tag{8.26}$$

and

$$f(1-\sqrt{3}) = -2 = f_0 \mathcal{I} + f_1(1-\sqrt{3}), \quad f(1+\sqrt{3}) = -2 = f_0 \mathcal{I} + f_1(1+\sqrt{3}) \tag{8.27}$$

whose solution is $f_0 = -2$ and $f_1 = 0$. Therefore

$$f(A) = -2\mathcal{I}. \tag{8.28}$$

Therefore the only eigenvalue is $-2$. In the last case we have

$$a\mathcal{I} + \frac{b}{\sqrt{2}}(\sigma_1 + \sigma_3) = \begin{bmatrix} a + \frac{b}{\sqrt{2}} & \frac{b}{\sqrt{2}} \\ \frac{b}{\sqrt{2}} & a - \frac{b}{\sqrt{2}} \end{bmatrix} \tag{8.29}$$

whose eigenvalues are $\lambda_- = a - b$ and $\lambda_+ = a + b$; now

$$A = f(a\mathcal{I} + \frac{b}{\sqrt{2}}(\sigma_1 + \sigma_3)) = f_0 \mathcal{I} + f_1 \left[ a\mathcal{I} + \frac{b}{\sqrt{2}}(\sigma_1 + \sigma_3) \right], \tag{8.30}$$

and

$$f_- := f(\lambda_-) = f_0 + f_1(a-b), \quad f_+ := f(\lambda_+) = f_0 + f_1(a+b) \tag{8.31}$$

whose solution (for $a \neq 0, b \neq 0$)is

$$f_0 = \frac{a(f_- - f_+) + b(f_- + f_+)}{2b}, \quad f_1 = \frac{f_+ - f_-}{2b}. \tag{8.32}$$

In the end

$$A = (f_0 + af_1)\mathcal{I} + f_1 b\vec{n}\cdot\vec{\sigma} = \left[ \frac{a(f_- - f_+) + b(f_- + f_+) + af_+ - af_-}{2b} \right] \mathcal{I} + \left[ \frac{f_+ - f_-}{2} \right] \vec{n}\cdot\vec{\sigma}; \tag{8.33}$$

for $a = 0, b = 1, f_- = f_+ = 1$ we get

$$A = \mathcal{I}. \tag{8.34}$$

## 8.4 Matrix ODEs

Solve the following matrix first order ODEs:

1. $\frac{dx}{dt} = 3x - 4y, \frac{dy}{dt} = 4x - 7y, \quad x(0) = y(0) = 1;$
2. $\frac{dy}{dt} = 3\frac{dy}{dt} + \frac{1}{t}\frac{dy}{dt} - ty, \quad y(0) = 1$. Find the solution up to third order in $t$.

The general solution of a matrix ODE of the form

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}[\boldsymbol{x}(t) - \boldsymbol{b}] \tag{8.35}$$

is

$$\boldsymbol{x}(t) = \boldsymbol{b} + e^{\boldsymbol{A}t}[\boldsymbol{x}(0) - \boldsymbol{b}]. \tag{8.36}$$

In the first case

$$\boldsymbol{A} = \begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{8.37}$$

So we need to compute the exponential of a matrix; the eigenvalues of $\boldsymbol{A}$,

$$\lambda_1 = -5, \quad \lambda_2 = 1. \tag{8.38}$$

At this point we can compute the diagonal exponentiated matrix and return to the non-diagonal using the base changing matrix; however, using Caley-Hamilton theorem is easier:

$$e^{-5t} = f_0 - 5tf_1, \quad e^t = f_0 + tf_1; \tag{8.39}$$

the solution is

$$f_0 = \frac{e^{-5t}}{6} + \frac{5e^t}{6}, \quad f_1 = \frac{e^t}{6t} - \frac{e^{-5t}}{6t} \tag{8.40}$$

Now, the solution of the second case is

$$\boldsymbol{x}(t) = \frac{1}{6}\left(\left(e^{-5t} + 5e^t\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\frac{e^t - e^{-5t}}{t}\right)\begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix} t\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{8.41}$$

so

$$\begin{aligned} \boldsymbol{x}(t) &= \exp\left(\begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix} t\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4e^t/3 - e^{-5t}/3 & 2e^{-5t}/3 - 2e^t/3 \\ 2e^t/3 - 2e^{-5t}/3 & 4e^{-5t}/3 - e^t/3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \\ &= \begin{bmatrix} e^{-5t}/3 + 2e^t/3 \\ e^t/3 + 2e^{-5t}/3 \end{bmatrix}. \end{aligned} \tag{8.42}$$

In the second case we need to reduce to a system, let us write $\frac{dy}{dt} = tx$ than:

$$\frac{dy^2}{dt^2} = t\frac{dx}{dt} + x; \tag{8.43}$$

so

$$t\frac{dx}{dt} + x = 3tx + x - ty \Rightarrow \frac{dx}{dt} = 3x - y, \quad \frac{dy}{dt} = tx. \tag{8.44}$$

In matrix form

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ t & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{8.45}$$

with initial condition $\boldsymbol{x}(0) = \mathbf{1}$. Note that

$$\int_0^t \boldsymbol{A}(s)ds = \int_0^t \begin{bmatrix} 3 & -1 \\ s & 0 \end{bmatrix} ds = \begin{bmatrix} 3t & -t \\ \frac{t^2}{2} & 0 \end{bmatrix} \tag{8.46}$$

does not commute with $\boldsymbol{A}$. The solution is given by the T product

$$\boldsymbol{x}(t) = \mathrm{T}\left[exp\left(\int_0^t \boldsymbol{A}(s)ds\right)\right] \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{8.47}$$

Expanding up to second order we get

$$\boldsymbol{x}(t) = \left[\mathcal{I} + \int_0^t \boldsymbol{A}(s)ds + \int_0^t ds \int_0^s \boldsymbol{A}(s)\boldsymbol{A}(q)dq + ...\right] \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8.48}$$

where

$$\begin{aligned} \int_0^t \boldsymbol{A}(s)ds &= \int_0^t \begin{bmatrix} 3 & -1 \\ s & 0 \end{bmatrix} ds = \begin{bmatrix} 3t & -t \\ \frac{t^2}{2} & 0 \end{bmatrix}, \\ \int_0^t ds \int_0^s \boldsymbol{A}(s)\boldsymbol{A}(q)dq &= \int_0^t ds \int_0^s \begin{bmatrix} -q+9 & -3 \\ 3s & -s \end{bmatrix} dq = \int_0^t ds \begin{bmatrix} -\frac{s^2}{2} + 9s & -3s \\ 3s^2 & -s^2 \end{bmatrix} = \\ &= \begin{bmatrix} -\frac{t^3}{6} + \frac{9}{2}t^2 & -\frac{3}{2}t^2 \\ t^3 & -\frac{t^3}{3} \end{bmatrix} \end{aligned} \tag{8.49}$$

so the solution, up to third order in $t$ is,

$$\boldsymbol{x}(t) = \left(\begin{bmatrix} \frac{-t^3+27t^2+18t+6}{6} & \frac{-3t^2-2t}{2} \\ \frac{2t^3+t^2}{2} & \frac{-t^3+3}{3} \end{bmatrix} + ...\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-t^3+18t^2+12t+6}{6} \\ \frac{4t^3+3t^2+6}{6} \end{bmatrix} + ... \tag{8.50}$$

# 9 Abstract linear spaces

## 9.1 Lebesgue spaces

Determine for which $q$ the following functions belongs to $L^q(\mathbb{R}^n, \mathcal{L}^n)$:

- $u(x) = (1 + |x|)^{-\frac{n}{p}}$;
- $u(x) = \chi_{B_1(0)}|x|^{-\frac{n}{p}}$

To belongs to $L^q(\mathbb{R}^n, \mathcal{L}^n)$ with $q \in [1, +\infty]$ we need that

$$\int |f(x)|^q d^n x < +\infty \quad for\ q \in [1, +\infty) \tag{9.1}$$

and

$$sup\{C \mid |f(x)| < C, q.o.\} < +\infty \quad for\ q = +\infty[1, +\infty). \tag{9.2}$$

For the first case we have

$$\int_{\mathbb{R}^n} \left| \frac{1}{1 + |x|^{\frac{n}{p}}} \right|^q d^n x = \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^{\frac{n}{p}})^q} d^n x = \Omega \int_0^{+\infty} \frac{1}{(1 + \rho^{\frac{n}{p}})^q} \rho^{n-1} d\rho \tag{9.3}$$

this function explodes at infinity, where the function behaves as

$$u(x) = \frac{1}{\rho^{\frac{qn}{p}}}, \tag{9.4}$$

so we need to impose that at infinity the integrals behave better that a logarithm, namely

$$n - 1 - \frac{nq}{p} < -1 \Rightarrow p < q. \tag{9.5}$$

For the second case we have

$$\int_{\mathbb{R}^n} \left| \frac{1}{|x|^{\frac{n}{p}}} \right|^q d^n x = \Omega \int_0^1 \frac{1}{(\rho^{\frac{n}{p}})^q} \rho^{n-1} d\rho; \tag{9.6}$$

in this case we have problems at zero, so again function need to behaves better than logarithm at zero, so

$$n - 1 - \frac{nq}{p} > -1 \Rightarrow p > q. \tag{9.7}$$

# 10 Eigenvalue problems in infinite dimensional spaces

## 10.1 Resolvent operator for finite-dimensional matrices

Using the resolvent operator find:

- $A^{\frac{1}{2}}$ with $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ and using the determination with the branch cut on the imaginary negative axis

Recall that the resolvant operator is defined as

$$R(z; T) = (z\mathcal{I} - T)^{-1} \tag{10.1}$$

where $T$ is an operator; the set of complex numbers $z$ such that the operator is invertible is the complementary set of the spectrum of the operator. Moreover from the resolvant operator we can compute, using residues, the projection operators entering in the spectral decomposition. In general we can compute the projectors using

$$\begin{aligned} P_{\lambda_k} &= \frac{\prod_{n\neq k}(T-\lambda_n\mathcal{I})}{\prod_{n\neq k}(\lambda_k-\lambda_n)} \quad \textit{non degeneracy case;} \\ P_{\lambda_{k_j}} &= \frac{\prod_{\lambda_{k_n}\neq\lambda_{k_j}}(T-\lambda_{k_n}\mathcal{I})}{\prod_{\lambda_{k_n}\neq\lambda_{k_j}}(\lambda_{k_j}-\lambda_{k_n})} \quad \textit{degeneracy case.} \end{aligned} \tag{10.2}$$

The resolvant operator for the matrix $A$ is given by (using the definition)

$$R(A;z) = \begin{bmatrix} 1+z & -1 & 0 \\ 0 & 1+z & -1 \\ 0 & 0 & -4-z \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{z+1} & \frac{1}{(z+1)^2} & \frac{1}{(z+1)^2(z-4)} \\ 0 & \frac{1}{z+1} & \frac{1}{(z+1)(z+4)} \\ 0 & 0 & \frac{1}{z-4} \end{bmatrix}. \tag{10.3}$$

The eigenvalues of $A$ are $\lambda_{1_1} = \lambda_{1_2} = -1$ and $\lambda_3 = 4$. The projection operators associated to $\lambda_1$ and $\lambda_3$ are computed calculating the residues of the function in the entrees of the resolvent operator for $z = \lambda_1$ and $z = \lambda_2$, we get

$$P_{\lambda_{1_1}} = \begin{bmatrix} 1 & 0 & -\frac{1}{25} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}, \quad P_{\lambda_3} = \begin{bmatrix} 0 & 0 & \frac{1}{25} \\ 0 & 0 & \frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix}. \tag{10.4}$$

To find the other projector operator associated to the degenerate eigenvalues $-1$ (namely $\lambda_{1_2}$) we use the fact that

$$P_{\lambda_{1_1}} + P_{\lambda_{1_2}} + P_{\lambda_3} = \mathcal{I} \Rightarrow P_{\lambda_{1_2}} = \begin{bmatrix} 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{10.5}$$

Using the spectral decomposition we know that

$$f(A) = f(\lambda_{1_1})P_{\lambda_{1_1}} + f'(\lambda_{1_2})P_{\lambda_{1_2}} + f(\lambda_3)P_{\lambda_3} \tag{10.6}$$

so using that for our determination $(-1)^{\frac{1}{2}} = -i$ we get

$$A^{\frac{1}{2}} = -iP_{\lambda_{1_1}} + \frac{i}{2}P_{\lambda_{1_2}} + 2P_{\lambda_3} = \begin{bmatrix} -i & \frac{i}{2} & \frac{i}{25} - \frac{i}{10} + \frac{2}{25} \\ 0 & i & \frac{i}{5} + \frac{2}{5} \\ 0 & 0 & 2 \end{bmatrix}. \tag{10.7}$$

## 10.2 Spectrum of an operator

Given the following operators find

- the punctual spectrum and the eigenfunctions of $T = -i\frac{d}{dx} + e^x$ with $\mathcal{D}(T) := \{f \in L^2[0,1], f' \in L^2[0,1] \mid f(0) = f(1)\}$;

- the punctual spectrum and the eigenfunctions of $T = \frac{d}{dx} + \frac{1}{xln(x)}$ with $\mathcal{D}(T) := \{f \in L^2[e, e^2], f' \in L^2[e, e^2] \mid f(e) = f(e^2)\}$;
- the punctual spectrum of the Fourier transform operator on $L^2$, $\mathcal{F}$, knowing that $\mathcal{F}^2 = R$ where $R$ is a reflection (involutionary) operator.

For first case we need to solve the equation

$$\Big( -i\frac{d}{dx} + e^x \Big) f = \lambda f \tag{10.8}$$

whose solution is given by

$$f(x) = ce^{-i\int_0^x (e^{x'} - \lambda)dx'} = ce^{-i(e^x - 1 - \lambda x)}; \tag{10.9}$$

imposing the boundary condition we get $f(1) = f(0)$

$$f(1) = ce^{-i(e-1-\lambda)} = f(0) = c \Rightarrow e^{-i(e-1-\lambda)} = 1, \tag{10.10}$$

so we require that

$$\lambda_k = e - 1 + 2k\pi \quad k \in \mathbb{Z}. \tag{10.11}$$

In the and

$$\begin{aligned} &eigenvalues \Rightarrow \lambda_k = e - 1 + 2k\pi \quad k \in \mathbb{Z}; \\ &eigenfunctions \Rightarrow f_k(x) = f(0)e^{-i(e^x - 1 - \lambda_k x)}. \end{aligned} \tag{10.12}$$

For the second case we need to solve the equation

$$\Big( \frac{d}{dx} + \frac{1}{xln(x)} \Big) f = \lambda f \tag{10.13}$$

whose solution is given by

$$f(x) = ce^{-\int_e^x (\frac{1}{x'ln(x')} - \lambda)dx'} = ce^{-ln(ln(x)) + \lambda(x-e)} = c\frac{e^{\lambda(x-e)}}{ln(x)}; \tag{10.14}$$

imposing the boundary condition we get

$$f(e^2) = c\frac{e^{\lambda(e^2-e)}}{2} = f(e) = c \Rightarrow \frac{e^{\lambda(e^2-e)}}{2} = 1, \tag{10.15}$$

so we require that

$$\lambda_k = \frac{ln(2) + 2k\pi i}{e^2 - e}. \tag{10.16}$$

In the and

$$\begin{aligned} &eigenvalues \Rightarrow \lambda_k = \frac{ln(2) + 2k\pi i}{e^2 - e} \quad k \in \mathbb{Z}; \\ &eigenfunctions \Rightarrow f_k(x) = f(0)\frac{e^{\lambda_k(x-e)}}{ln(x)}. \end{aligned} \tag{10.17}$$

For the last case we only need to know that, being $R$ an involution

$$R^2 = \mathcal{I} \Rightarrow \mathcal{F}^4 = \mathcal{I}; \tag{10.18}$$

since for Caley-Hamilton theorem a linear operator satisfy it characteristic polynomial, this means that

$$\lambda_k^4 = 1 \Rightarrow \lambda_k = i^k \quad k = 0, 1, 2, 3. \tag{10.19}$$

# 11 Fourier transformation and series

## 11.1 Fourier series

Compute the Fuorier series of the following functions and use the result to sum the indicated series:

- $f(x) = e^{i\frac{x}{2}}, \quad -\pi < x < \pi, \quad \sum_{-\infty}^{+\infty} \frac{(-1)^n}{n-\frac{1}{2}};$
- $g(x) = x(x-2\pi), \quad 0 < x < 2\pi, \quad \sum_1^{\infty} \frac{-(1)^n}{n^2}.$

To compute the Fourier series of a generic function $h(x)$ we need to compute

$$\begin{aligned} a_n &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} h(x) cos\left(\frac{2\pi}{T}nx\right)dx; \\ b_n &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} h(x) sin\left(\frac{2\pi}{T}nx\right)dx; \end{aligned} \tag{11.1}$$

and the Fourier series will be given by

$$h(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi}{T}nx\right) + b_n \sin\left(\frac{2\pi}{T}nx\right)\right]; \tag{11.2}$$

or equivalently

$$h(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i2\pi nx}{T}}, \tag{11.3}$$

with

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} h(x) e^{\frac{-i2\pi nx}{T}} dx. \tag{11.4}$$

Let us start with the first case; we have to compute

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\frac{x}{2}} e^{-inx} dx = \frac{1}{2\pi[\frac{i}{2} - in]} e^{x[\frac{i}{2}-in]}\Big|_{-\pi}^{\pi} = \frac{1}{2\pi[\frac{i}{2} - in]}[e^{\frac{i\pi}{2} - in\pi} - e^{-\frac{i\pi}{2} + in\pi}] = \\ &= \frac{2isin((2n-1)\frac{\pi}{2})}{2\pi i[\frac{1}{2} - n]} = -\frac{(-1)^n}{\pi[\frac{1}{2} - n]}; \end{aligned} \tag{11.5}$$

so we have

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\pi[n-\frac{1}{2}]} e^{inx}. \tag{11.6}$$

We note that for $x = 0$ that the following relation holds (since the function in continuous in $x = 0$ the series converges in that point)

$$1 = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\pi[n-\frac{1}{2}]} \Rightarrow \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{[n-\frac{1}{2}]} = \pi. \tag{11.7}$$

In the second case we have to compute

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} [x(x-2\pi)] cos\Big(nx\Big) dx; \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} [x(x-2\pi)] sin\Big(nx\Big) dx; \end{aligned} \tag{11.8}$$

integrating by parts we get

$$a_0 = \frac{2\pi^2}{3}, \quad a_{n>0} = (-1)^n \frac{4}{n^2}, \quad b_n = (-1)^n \frac{4\pi}{n}, \tag{11.9}$$

so

$$g(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[ (-1)^n \frac{4}{n^2} \cos(nx) + (-1)^n \frac{4\pi}{n} \sin(nx) \right]. \tag{11.10}$$

we note that in $x = 0$ (the series converge in that point because the function is continuous) we have

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}. \tag{11.11}$$

## 11.2 Computing Fourier transformations

Compute the following Fourier transformation of the following functions:

- $f(x) = e^{-|x|}$;
- $\phi(\vec{q}) = \frac{1}{|\vec{q}|^2 + \left(\frac{Mc}{\hbar}\right)^2}$; with $\vec{q} \in \mathbb{R}^n$ (do the antitransform express it as an integral from 0 to $+\infty$);
- $h(\vec{x}) = e^{-a|\vec{x}|^2}$ with $\vec{x} \in \mathbb{R}^7$.

Let us use the definition of the Fourier transform with the $\frac{1}{2\pi}$ factor in its inverse. For the first case we have

$$\begin{aligned} \hat{f}(p) &= \int_{-\infty}^{+\infty} e^{-|x|} e^{-ipx} dx = \int_{-\infty}^{0} e^{x} e^{-ipx} dx + \int_{0}^{+\infty} e^{-x} e^{-ipx} dx = \\ &= \frac{e^{x-ipx}|_{-\infty}^{0}}{1-ip} + \frac{e^{-x-ipx}|_{0}^{\infty}}{-1-ip} = \frac{1}{1-ip} + \frac{1}{1+ip} = \frac{2}{1+p^2}. \end{aligned} \tag{11.12}$$

For the second case we have

$$\phi(\vec{r}) = \int \frac{d\vec{q}}{(2\pi)^n} \frac{e^{i\vec{q}\cdot\vec{r}}}{|\vec{q}|^2 + \left(\frac{Mc}{\hbar}\right)^2} \tag{11.13}$$

we note that

$$\int_0^{\infty} d\beta e^{-\beta(|\vec{q}|^2+k^2)} = -\frac{1}{|\vec{q}|^2+k^2} e^{-\beta(|\vec{q}|^2+k^2)}\Big|_0^{\infty} = \frac{1}{|\vec{q}|^2+k^2},$$

so, fixing $k = \frac{Mc}{\hbar}$, we get

$$\phi(\vec{r}) = \frac{1}{(2\pi)^n} \int d\vec{q} \int_0^{\infty} d\beta e^{i\vec{q}\cdot\vec{r}-\beta(q^2+k^2)} = \frac{1}{(2\pi)^n} \int_0^{\infty} d\beta \prod_{i=1}^{n} \int_{-\infty}^{+\infty} dq_i e^{iq_i r_i - \beta(q_i^2+k^2)}; \tag{11.14}$$

using the gaussian integral

$$\int_{-\infty}^{+\infty} dqae^{-bq^2+cq+d} = a\sqrt{\frac{\pi}{b}} e^{\frac{c^2}{4b}+d},$$

where, for our case, $a = 1$, $b = \beta$, $c = ir_i$ e $d = -\beta k^2$, we get

$$\phi(\vec{r}) = \frac{1}{(2\pi)^n} \int_0^{\infty} \prod_{i=1}^{n} \left[\sqrt{\frac{\pi}{\beta}} e^{-\frac{r_i^2}{4\beta} - \beta\left(\frac{Mc}{\hbar}\right)^2}\right] = \frac{\pi^{\frac{n}{2}}}{(2\pi)^n} \int_0^{\infty} d\beta \frac{e^{-\frac{r^2}{4\beta} - \beta\left(\frac{Mc}{\hbar}\right)^2}}{\beta^{\frac{n}{2}}}. \tag{11.15}$$

In the last case we have the product of seven Fourier integrals of the one-dimensional gaussian

$$\tilde{h}(\vec{p}) = \left(\int_{-\infty}^{+\infty} e^{-ax_1^2} e^{-ip_1 x_1} dx_1\right) \ldots \left(\int_{-\infty}^{+\infty} e^{-ax_7^2} e^{-ip_7 x_7} dx_7\right); \tag{11.16}$$

now, starting from

$$\int_{-\infty}^{+\infty} e^{-t_i^2} dx_i = \sqrt{\pi} \tag{11.17}$$

using the change of variables $t_i = \sqrt{a}x_i + \frac{p_i}{2\sqrt{a}}$ we get

$$\int_{-\infty}^{+\infty} e^{-ax_i^2} e^{-p_i x_i} dx_i = \sqrt{\frac{\pi}{a}} e^{\frac{p_i^2}{4a}}. \tag{11.18}$$

At this point we note that if we substitute $p_i \to ip_i$, equation 11.18 is extended to holomophic functions that, since they coincide on the real axis, must coincide in all the complex plane. Therefore

$$\int_{-\infty}^{+\infty} e^{-ax_i^2} e^{-ip_i x_i} dx_i = \sqrt{\frac{\pi}{a}} e^{-\frac{p_i^2}{4a}}; \tag{11.19}$$

in the end

$$\tilde{h}(\vec{p}) = \left(\frac{\pi}{a}\right)^{\frac{7}{2}} e^{-\frac{|\vec{p}|^2}{4a}} \tag{11.20}$$

## 11.3 Fourier transformation and ODEs

Find a particular solution of following ODEs using Fourier transform:

- $\ddot{z} + \gamma\dot{z} + \omega_0^2 z = Ae^{i\Omega t}$;
- $\ddot{x} - m^2 x = J$;
- $3x''' + x' + x = t$

We know that the Fourier transform of $\dot{x}(t)$ is $-ik\tilde{x}(k)$ where $\tilde{x}(k)$ is the Fourier transform of $x(t)$ (we adopt this convention). Therefore we have in Fourier transform we have

$$\begin{aligned}
&(-k^2 + i\gamma k + \omega_0^2)\tilde{z}(k) = A\int_{-\infty}^{+\infty} e^{i\Omega t}e^{-ikt}dt = 2\pi A\delta(\Omega - k);\\
&(-k^2 - m^2)\tilde{x}(k) = J\int_{-\infty}^{+\infty} e^{-ikt}dt = 2\pi J\delta(-k);\\
&(3ik^3 - ik + 1)\tilde{x}(k) = \int_{-\infty}^{+\infty} te^{-ikt}dt = i2\pi\delta'(k)
\end{aligned} \tag{11.21}$$

So the solutions in Fourier space are

$$\begin{aligned}
\tilde{z}(k) &= \frac{2\pi A\delta(\Omega - k)}{(-k^2 + i\gamma k + \omega_0^2)};\\
\tilde{x}(k) &= \frac{2\pi J\delta(-k)}{-k^2 - m^2};\\
\tilde{x}(k) &= \frac{i2\pi\delta'(k)}{3ik^3 - ik + 1}
\end{aligned} \tag{11.22}$$

while the solutions in real space are

$$\begin{aligned}
z(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2\pi A\delta(\Omega - k)}{(-k^2 + i\gamma k + \omega_0^2)}e^{ikt}dk = \frac{Ae^{i\Omega t}}{-\Omega^2 + i\gamma\Omega + \omega_0^2};\\
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2\pi J\delta(-k)}{-k^2 - m^2}e^{ikt}dk = -\frac{J}{m^2};\\
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{i2\pi\delta'(k)}{3ik^3 - ik + 1}e^{ikt}dk = t - 1
\end{aligned} \tag{11.23}$$

# 12 Integral equations

## 12.1 ODEs and Volterra integral equations

Given the following ODEs rewrite them as integral equations:

1. $y''(x) + xy'(x) + y = 0, \quad y(0) = 1, y'(0) = 0;$

2. $y''(x) + y = 0, \quad y(0) = 0, \; y'(0) = 1;$

3. $y''(x) + y = cos(x), \quad y(0) = y'(0) = 0;$

The general procedure is to put $\phi(x) = y^{(n)}$ where $n$ is the maximal order derivative appearing in the ODE. Then we star to integrate using the fundamental theorem of calculus and the identity

$$\underbrace{\int_{x_0}^{x} dx ... \int_{x_0}^{x} dx}_{n-times} f(x) \frac{1}{(n-1)!} \int_{x_0}^{x} (x-z)^{n-1} f(z) dz \tag{12.1}$$

and the boundary conditions. Using this procedure we can transform an ODE in a Volterra integral equation of the II kind.

In all cases we put $\phi(x) = y''(x)$. In the first case we have

$$y'(x) = \int_0^x \phi(t)dt + y'(0) = \int_0^x \phi(t)dt, \tag{12.2}$$

so

$$y(x) = \int_0^x dt \int_0^x dt \phi(t) + y(0) = \int_0^x (x-t)\phi(t)dt + 1. \tag{12.3}$$

Inserting in the original ODE we get

$$\phi(x) + x \int_0^x \phi(t)dt + \int_0^x (x-t)\phi(t)dt + 1 = 0; \tag{12.4}$$

this is a Volterra integral equation of the II kind with $K(x,t) = 2x - t, \; \lambda = -1, \; f(x) = -1$.

In the second case we have

$$y'(x) = \int_0^x \phi(t)dt + y'(0) = \int_0^x \phi(t)dt + 1, \tag{12.5}$$

so

$$y(x) = \int_0^x dt \int_0^x dt \phi(t) + \int_0^x dt + y(0) = \int_0^x (x-t)\phi(t)dt + x. \tag{12.6}$$

Inserting in the original ODE we get

$$\phi(x) + \int_0^x (x-t)\phi(t)dt + x = 0; \tag{12.7}$$

this is a Volterra integral equation of the II kind with $K(x,t) = x - t, \; \lambda = -1, \; f(x) = -x$.

In the last case we have

$$y'(x) = \int_0^x \phi(t)dt + y'(0) = \int_0^x \phi(t)dt, \tag{12.8}$$

so

$$y(x) = \int_0^x dt \int_0^x dt\phi(t) + y(0) = \int_0^x (x-t)\phi(t)dt. \tag{12.9}$$

Inserting in the original ODE we get

$$\phi(x) + \int_0^x (x-t)\phi(t)dt + cos(x) = 0; \tag{12.10}$$

this is a Volterra integral equation of the II kind with $K(x,t) = -x+t,\ \lambda = 1,\ f(x) = cos(x)$.

## 12.2 Volterra integral equations of II kind

Solve the following Volterra integral equations of the II kind using the iterated kernel method:

1. $\phi(x) = 1 + \int_0^x e^{x-t+m}\phi(t)dt;$
2. $\phi(x) = x + \sqrt{2}\int_0^x \phi(t)dt.$

Given the general Volterra integral equation of the II kind

$$\phi(x) = f(x) + \lambda \int_0^x K(x,t)\phi(t)dt \tag{12.11}$$

the solution can be expressed in term of the risolvent operator as

$$\phi(x) = f(x) + \lambda \int_0^x R(x,t,\lambda)f(t)dt \tag{12.12}$$

where

$$R(x,t,\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t) \tag{12.13}$$

where the iterated kernels are defined as

$$K_1(x,t) = K(x,t), \quad K_{n+1}(x,t) = \int_t^x K(x,z)K_n(z,t)dz. \tag{12.14}$$

In the first case we have $K_1(x,t) = e^{x-t}$ so the iterated kernels are

$$\begin{aligned}
K_2(x,t) &= \int_t^x e^{x-z}e^{z-t}dz = e^{x-t}(x-t); \\
K_3(x,t) &= \int_t^x (z-t)e^{x-z}e^{z-t}dz = e^{x-t}\int_t^x (z-t)dz = e^{x-t}\int_0^{x-t} ydy = \frac{(x-t)^2}{2}e^{x-t}; \\
&\vdots \\
K_n(x,t) &= \int_t^x \frac{(z-t)^{n-2}}{(n-2)!}e^{x-z}e^{z-t}dz = \frac{(x-t)^{n-1}}{(n-1)!}e^{x-t}.
\end{aligned} \tag{12.15}$$

The risolvent operator is given by

$$R(x,t,\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n (z-t)^n}{n!} e^{x-t} = e^{\lambda(x-t)} e^{x-t} = e^{(\lambda+1)(x-t)}, \tag{12.16}$$

therefore the solution is simply

$$\phi(x) = 1 + \lambda \int_0^x e^{(\lambda+1)(x-t)} dt = 1 + \lambda e^{(\lambda+1)x} \int_0^x e^{-(\lambda+1)t} dt = 1 - \frac{\lambda}{\lambda+1} + \frac{e^{(\lambda+1)x}}{\lambda+1}. \tag{12.17}$$

In the end

$$\phi(x) = 1 - \frac{e^m}{e^m+1} + \frac{e^{(e^m+1)x}}{e^m+1}. \tag{12.18}$$

In the second case we have $K_1(x,t) = 1$ so the iterated kernels are

$$\begin{aligned} K_2(x,t) &= \int_t^x dz = x - t; \\ K_3(x,t) &= \int_t^x (z-t) dz = \int_0^{x-t} y dy = \frac{(x-t)^2}{2}; \\ &\vdots \\ K_n(x,t) &= \int_t^x \frac{(z-t)^{n-2}}{(n-2)!} dz = \frac{(x-t)^{n-1}}{(n-1)!}. \end{aligned} \tag{12.19}$$

The risolvent operator is given by

$$R(x,t,\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n (z-t)^n}{n!} = e^{\lambda(x-t)}, \tag{12.20}$$

therefore the solution is simply

$$\phi(x) = x + \lambda \int_0^x t e^{\lambda(x-t)} dt = x + \lambda e^{\lambda x} \Big[ -\frac{x}{\lambda} e^{-\lambda x} - \int_0^x e^{-\lambda t} dt \Big] = 1 - e^{\lambda x}. \tag{12.21}$$

In the and

$$\phi(x) = 1 - e^{\sqrt{2}x}. \tag{12.22}$$

## 12.3 Fredholm integral equations of II kind

Solve the following Fredholm integral equations of the II kind using the Fredholm determinats method:

1. $\phi(x) - \lambda \int_0^1 x e^t \phi(t) dt = e^{-x}, \quad (\lambda \neq 1);$
2. $\phi(x) - \lambda \int_0^1 (x - 2t) \phi(t) dt = 0.$

The general Fredholm integral equation of the II kind

$$\phi(x) - \lambda \int_a^b K(x,t)\phi(t)dt = f(x) \tag{12.23}$$

can be resolved using resolvent formalism

$$\phi(x) = f(x) + \lambda \int_a^b R(x,t,\lambda) f(t)dt. \tag{12.24}$$

The risolvent operator can be computed in terms of the Fredhoml determinants:

$$R(x,t,\lambda) = \frac{FD(x,t,\lambda)}{FD(\lambda)} \tag{12.25}$$

where

$$\begin{aligned} FD(x,t,\lambda) &= K(x,t) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \lambda^n B_n(x,t); \\ FD(\lambda) &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \lambda^n C_n; \end{aligned} \tag{12.26}$$

with

$$B_n(x,t) = \underbrace{\int_a^b dt_1 ... \int_a^b dt_n}_{n-times} \left| \begin{bmatrix} K(x,t) & \dots & K(x,t_n) \\ \vdots & \dots & \vdots \\ K(t_n,t) & \dots & K(t_n,t_n) \end{bmatrix} \right|, \quad B_0(x,t) = K(x,t) \tag{12.27}$$

and

$$C_n = \underbrace{\int_a^b dt_1 ... \int_a^b dt_n}_{n-times} \left| \begin{bmatrix} K(t_1,t) & \dots & K(t_1,t_n) \\ \vdots & \dots & \vdots \\ K(t_n,t) & \dots & K(t_n,t_n) \end{bmatrix} \right|, \quad C_0 = 1. \tag{12.28}$$

In general these determinants are difficult to compute but we have the following reletions

$$B_n(x,t) = C_n K(x,t) - n \int_a^b K(x,s) B_{n-1}(s,t) ds; \quad C_n = \int_a^b B_{n-1}(s,s) ds. \tag{12.29}$$

In the fir case we have $K(x,t) = xe^t$ and $B_0(x,t) = xe^t$. Let us compute the Fredholm determinants using their definition. First of all we have

$$B_1(x,t) = \int_0^1 \left| \begin{bmatrix} xe^t & xe^{t_1} \\ t_1 e^t & t_1 e^{t_1} \end{bmatrix} \right| dt_1 = 0, \quad B_n(x,t) = 0 \; \forall n; \tag{12.30}$$

and

$$C_1 = \int_0^1 t_1 e^{t_1} dt_1 = \int_0^1 \frac{\partial}{\partial\alpha} e^{\alpha t_1} dt_1 \Big|_{\alpha=1} = \frac{\partial}{\partial\alpha} \int_0^1 e^{\alpha t_1} dt_1 \Big|_{\alpha=1} = \frac{\partial}{\partial\alpha}\left[\frac{1}{\alpha}\Big(e^\alpha - 1\Big)\right]\Big|_{\alpha=1} =$$

$$= \left( -\frac{1}{\alpha^2}\Big[e^\alpha - 1\Big] + \frac{1}{\alpha}e^\alpha \right)\Big|_{\alpha=1} = 1;$$

$$C_2 = \int_0^1 \int_0^1 \left| \begin{bmatrix} t_1 e^{t_1} & t_1 e^{t_2} \\ t_1 e^{t_2} & t_2 e^{t_2} \end{bmatrix} \right| dt_1 dt_2 = 0, \quad C_n = 0 \forall n; \tag{12.31}$$

therefore

$$FD(x,t,\lambda) = K(x,t) = xe^t, \quad FD(\lambda) = 1 - \lambda. \tag{12.32}$$

The risolvent operator is therefore given by

$$R(x,t,\lambda) = \frac{xe^t}{1-\lambda}, \tag{12.33}$$

and the solution of the integral equation is given by

$$\phi(x) = e^{-x} + \lambda \int_0^1 \frac{xe^t}{1-\lambda} e^{-t} dt = e^{-x} + x\frac{\lambda}{1-\lambda}. \tag{12.34}$$

In the second case we use the recursive formula to compute the Fredholm determinant. We have $C0 = 1,\ B_0(x,t) = x - 2t$ so

$$C_1 = \int_0^1 -s ds = -\frac{1}{2},\ B_1(x,t) = -\frac{1}{2}(x-2t) - \int_0^1 (x-2s)(s-2t) ds = -x-t+2xt+\frac{2}{3}, \tag{12.35}$$

then

$$C_2 = \int_0^1 (-2s+2s^2+\frac{2}{3}) ds = \frac{1}{3},\ B_2(x,t) = \frac{1}{3}(x-2t) - 2\int_0^1 (x-2s)(-s-t-2st+\frac{2}{3}) ds = 0, \tag{12.36}$$

then

$$C_n = B_n(x,t) = 0 \forall n > 2. \tag{12.37}$$

The risolven opertor is

$$R(x,t,\lambda) = \frac{x - 2t + (x + t - 2xt - \frac{2}{3})\lambda}{1 + \frac{\lambda}{2} + \frac{\lambda^2}{6}} \tag{12.38}$$

and the solution of the integral equation is

$$\phi(x) = \lambda \int_0^1 \frac{x - 2t + (x + t - 2xt - \frac{2}{3})\lambda}{1 + \frac{\lambda}{2} + \frac{\lambda^2}{6}} dt = \frac{\lambda}{1 + \frac{\lambda}{2} + \frac{\lambda^2}{6}} \left[ x - 1 + \lambda\Big( x + \frac{1}{2} - x - \frac{2}{3} \Big) \right]. \tag{12.39}$$

# 13 Green function and Sturm-Liuville operators

## 13.1 Find Green functions

Given the following Cauchy problems find the Green function of the operator:

1. $y''''(x) = 0, \quad y(0) = y'(0) = 0, y(1) = y'(1) = 0;$
2. $y''(x) + k^2 y(x) = 0, \quad y(0) = y(1) = 0;$

The Green function of a boundary-value problem for a differential operator

$$D[y] = p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + ... + p_n(x)y = 0 \tag{13.1}$$

is defined as the function $G(x,t)$ defined for $a < t < b$ where $x \in [a,b]$ and given by the following equation

$$D[G(x,t)] = \delta(x-t). \tag{13.2}$$

It has to satisfy that it is continuous up to order $(n-2)$th derivative inclusive (for an $n$-order differential operator) while the $(n-1)$th derivative has a jump for $x = t$ equal to $\frac{1}{p_0(t)}$. Moreover it has to satisfy the boundary conditions. This means that in general the construction of the Green function is a replica of the tool used to construct weak solution of ODEs. However, for the specific class of Sturm-Liuville operators, namely

$$(p(x)y'(x))' + q(x)y(x) = 0, \quad y(a) = A, \; y(b) = B, \tag{13.3}$$

we have a nice formula

$$G(x,t) = \frac{1}{p(t)W(t)}[y_1(x)y_2(t)\Theta(t-x) + y_2(x)y_1(t)\Theta(x-t)] \tag{13.4}$$

where $y_1(x)$ and $y_2(x)$ are two linear independent solutions such that

$$y_1(a) = A, \; y_1(b) \neq B, \quad y_2(a) \neq A, \; y_2(b) = B \tag{13.5}$$

and W(x) is the Wronskian determinant.

Let us start from the first equation. This is linear and has as set of independent solutions $y_1(x) = 1, \; y_2(x) = x, \; y_3(x) = x^2, \; y_4(x) = x^3$; therefore the general solution is

$$y(x) = A + Bx + Cx^2 + Dx^3. \tag{13.6}$$

Imposing the boundary condition we get the relations

$$A = 0, \; B = 0, \; A + B + C + D = 0, \; B + 2C + 3D = 0 \tag{13.7}$$

whose solution is $A = B = C = D = 0$. The problem admits only the trivial solution and so the Green function is unique (this is a theorem: if a boundary-value differential problem admits only the trivial solution, the Green function of the operator is unique).Let us write the Green function as ($x \in [0,1]$)

$$G(x-t) = A_1+B_1x+C_1x^2+D_1x^3\Theta(t-x)+A_2+B_2x+C_2x^2+D_2x^3\Theta(x-t), \quad (13.8)$$

Imposing the continuity up to the third derivative excluded together with the jump on the third derivative at $x = t$, we get

$$\begin{aligned}
&A_2 - A_1 + (B_2 - B_1)t + (C_2 - C_1)t^2 + (D_2 - D_1)t^3 + = 0;\\
&B_2 - B_1 + 2(C_2 - C_1)t + 3(D_2 - D_1)t^2 = 0;\\
&2(C_2 - C_1) + 6(D_2 - D_1)t = 0;\\
&6(D_2 - D_1) = 1;
\end{aligned} \quad (13.9)$$

using, a t this point, that the Green function has to satisfy the boundary condition,namely

$$A_1 = 0,\ B_1 = 0,\ A_2 + B_2 + C_2 + D_2 = 0,\ B_2 + 2C_2 + 3D_2 = 0, \quad (13.10)$$

we can solve to find

$$\begin{aligned}
&A_1 = 0,\ B_1 = 0,\ C_1 = \frac{t}{2} - t^2 + \frac{t^3}{2},\ D_1 = -\frac{1}{6} + \frac{t^2}{2} - \frac{t^3}{6};\\
&A_2 = -\frac{t^3}{6},\ B_2 = \frac{t^2}{2},\ C_2 = -t^2 + \frac{t^3}{3},\ D_2 = \frac{t^2}{2} - \frac{t^3}{3}.
\end{aligned} \quad (13.11)$$

Plugging back we find the Green function.

Second case is simpler since this is a Sturm-Liuville operator. We may note that $y_1(x) = sin(kx)$ satisfy $y_1(0) = 0$ while $y_2(x) = sin((k-1)x)$ satisfy $y_2(1) = 0$ and they are linearly independent. The wroskian determinant is given by $W(x) = ksin(k)$ so

$$G(x,t) = \frac{1}{ksin(k)}[sin((k-1)t)sin(kx)\Theta(t-x) + (t \leftrightarrow s)]. \quad (13.12)$$

## 13.2 Solution of ODEs with Green function

Find the solution of the following ODEs using the Green function:

1. $y''(x) - y(x) = x, \quad y(0) = y(1) = 0;$
2. $y''(x) + y(x) = x, \quad y(0) = y\left(\frac{\pi}{2}\right) = 0;$
3. $y''(x) + \lambda y(x) = x, \quad y(0) = y\left(\frac{\pi}{2}\right) = 0.$

There is a theorem according to which if the boundary-vale problem $D[y] = 0$ with a set of boundary condition $B$ has Green function $G(x,t)$ the solution of the boundary-value problem $D[y] = f(x)$ with the same boundary conditions is given by

$$y(x) = \int_a^b G(x,t)f(t)dt \quad (13.13)$$

with $x \in [a,b]$. More generally if the boundary-value problem $D[y] = 0$ with a set of boundary condition $B$ has Green function $G(x,t)$ the solution of the boundary-value problem $D[y] = \lambda y(x) + f(x)$ with the same boundary conditions is given by the following Fredholm integral equation

$$y(x) = \lambda \int_a^b G(x,t)y(t)dt + \int_a^b G(x,t)f(t)dt \tag{13.14}$$

with $x \in [a,b]$.

Let us start from the first case. This is a Sturm-Liuville operator, let us find the Green function for the homogeneous problem. Solutions are $y_1(x) = sinh(x)$, which satisfy $y_1(0) = 0$, and $y_2(x) = sinh(x-1)$ which satisfy $y_2(1) = 0$. The wronskian determinant is

$$W(x) = \left| \begin{bmatrix} sinh(x) & sinh(x-1) \\ cosh(x) & cosh(x-1) \end{bmatrix} \right| = sinh(x)cosh(x-1) - cosh(x)sinh(x-1) = sinh(1) \tag{13.15}$$

where we used the sum and difference formulas for hyperbolic functions. The Green function is

$$G(x,t) = \frac{1}{sinh(1)}[sinh(x)sinh(t-1)\Theta(t-x) + (x \leftrightarrow t)]. \tag{13.16}$$

The solution of the original boundary-value problem is given by

$$\begin{aligned} y(x) &= \frac{1}{sinh(1)} \int_0^1 [sinh(x)sinh(t-1)\Theta(t-x) + (x \leftrightarrow t)]tdt = \\ &= \frac{sinh(x-1)}{sinh(1)} \int_0^x tsinh(t)dt + \frac{sinh(x)}{sinh(1)} \int_x^1 tsinh(t-1)dt, \end{aligned} \tag{13.17}$$

since

$$\int_0^x tsinh(t)dt = xcosh(x) - sinh(x), \quad \int_x^1 tsinh(t-1)dt = 1 - xcosh(x) + sinh(x-1), \tag{13.18}$$

we get

$$\begin{aligned} y(x) &= \frac{sinh(x-1)[xcosh(x) - sinh(x)] + sinh(x)[1 - xcosh(x-1) + sinh(x-1)]}{sinh(1)} = \\ &= \frac{sinh(x)}{sinh(1)} - x. \end{aligned} \tag{13.19}$$

In the second we have again a Sturm-Liuville operator; $y_1(x) = sin(x)$, such that $y_1(0) = 0$, and $y_2(x) = cos(x)$ such that $y_2\left(\frac{\pi}{2}\right) = 0$. The wronskian determinant is

$$W(x) = \left| \begin{bmatrix} sin(x) & cos(x) \\ cos(x) & -sin(x) \end{bmatrix} \right| = -sin^2(x) - cos^2(x) = -1 \tag{13.20}$$

and the Green function is given by

$$G(x,t) = -sin(x)cos(t)\Theta(t-x) - (x \leftrightarrow t). \tag{13.21}$$

The solution of the original boundary-value problem is given by

$$\begin{aligned} y(x) &= -\int_0^{\frac{\pi}{2}} [sin(x)cos(t)\Theta(t-x) + (x \leftrightarrow t)]tdt = \\ &= -cos(x)\int_0^x tsin(t)dt - sin(x)\int_x^{\frac{\pi}{2}} tcos(t)dt = \\ &= -cos(x)[sin(x) - xcos(x)] - sin(x)[cos(x) + xsin(x)] = \\ &= x[cos^2(x) - sin^2(x)] - 2cos(x)sin(x). \end{aligned} \tag{13.22}$$

The last case is more complicated. First of all let us determine the Green function for the homogeneous problem with $\lambda = 0$. This is a Sturm-Liuville operator and the two linearly independent solutions are $y_1(x) = x$, so $y_1(0) = 0$, and $y_2(x) = x - \frac{\pi}{2}$, so $y_2\left(\frac{\pi}{2}\right) = 0$. The wronskian determinant is easly computed to be $W(x) = \frac{\pi}{2}$; therefore the Green function is given by

$$G(x,t) = \frac{2}{\pi}\left[x\left(t - \frac{\pi}{2}\right)\Theta(t-x) + (x \leftrightarrow t)\right]. \tag{13.23}$$

The solution can be expressed as integral equation

$$\begin{aligned} y(x) &= -\lambda\int_0^{\frac{\pi}{2}} \frac{2}{\pi}\left[x\left(t - \frac{\pi}{2}\right)\Theta(t-x) + (x \leftrightarrow t)\right]y(t)dt + \\ &+ \int_0^{\frac{\pi}{2}} \frac{2}{\pi}\left[x\left(t - \frac{\pi}{2}\right)\Theta(t-x) + (x \leftrightarrow t)\right]tdt = \\ &= -\lambda\int_0^{\frac{\pi}{2}} \frac{2}{\pi}\left[x\left(t - \frac{\pi}{2}\right)\Theta(t-x) + (x \leftrightarrow t)\right]y(t)dt + \frac{x^3}{6} - \frac{\pi^2 x}{24}; \end{aligned} \tag{13.24}$$

we now have to solve this Fredholm integral equation. The solution is given in term of the risolvent operator

$$y(x) = \frac{x^3}{6} - \frac{\pi^2 x}{24} - \lambda\int_0^{\frac{\pi}{2}} \frac{FD(x,t,\lambda)}{FD(\lambda)}\left(\frac{t^3}{6} - \frac{\pi^2 t}{24}\right)dt; \tag{13.25}$$

# 14 PDEs

## 14.1 Wave equation

Solve the following problems for the wave equation:

1. $u_{tt} - c^2 u_{xx} = cos(x), \quad u(x,0) = sin(x),\ u_t(x,0) = 1 + x;$

2. $u_{tt} - u_{xx} = 0, \quad u(x,0) = 0,\ u_t(x,0) = 0,\ u(0,t) = 0,\ u(\pi,t) = 0,\ t \leq 0,\ x \in [0,\pi]$.

PDEs are not trivial to solve. In the case of nonhomogeneous wave equation we can solve a nonhomogeneous initial condition problem using d'Alambert and Duhamel formule

$$u(x,t) = u^A(x,t) + u^D(x,t) = \\ = \frac{1}{2}[g(x+ct) + g(x-ct]) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds \tag{14.1}$$

where $f(x,t)$ is the nonhomogeneous term while $g(x)$ and $h(x)$ are the initial conditions on the function and its derivative respectively. In the case of mixed boundary/initial condition and support in a limited region the solution can be found in term of Fourier series where the coefficients are constrained by the boundary/initial mixed conditions.

Let us start with the first problem. The solution is easily computed using d'Alambert and Duhamel fomrmule

$$\begin{aligned} u^A(x,t) &= \frac{1}{2}[sin(x+ct) + sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}(1+s)ds = \\ &= sin(x)cos(ct) + \frac{1}{2c}\left[s + \frac{s^2}{2}\right] = sin(x)cos(ct) + xt + t; \\ u^D(x,t) &= \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} cos(y)dyds = \\ &= \frac{1}{2c}\int_0^t [sin(x-c(t-s)) - sin(x+c(t-s))]ds = \frac{1}{c^2}(cos(x) - cos(x)cos(ct)). \end{aligned} \tag{14.2}$$

Therefore the solution is

$$u(x,t) = sin(x)cos(ct) + xt + x + \frac{1}{c^2}(cos(x) - cos(x)cos(ct)). \tag{14.3}$$

In the second case we have mixed conditions and we need to use Fourier series. We write

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t)con(nx) + b_n(t)sin(nx); \tag{14.4}$$

the boundary conditions fix the coefficients

$$u(0,t) = 0 \Rightarrow \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) = 0, \quad u(\pi,t) = 0 \Rightarrow \frac{a_0(t)}{2} - \sum_{n=1}^{\infty} a_n(t) = 0 \tag{14.5}$$

whose solution is $a_n(t) = 0 \forall n \in \mathbb{N}$. So we have

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)sin(nx) \tag{14.6}$$

and plugging back into the equation we get

$$\sum_{n=1}^{\infty} b''(t)sin(nx) + \sum_{n=1}^{\infty} b(t)n^2 sin(nx) = 0 \Rightarrow b''(t) + n^2 b(t) = 0; \tag{14.7}$$

this is an ODE whose solution is given by

$$b_n(t) = c_n sin(nt) + d_n cos(nt). \tag{14.8}$$

Coefficients are determined by initial conditions

$$u(x,0) = 1 \Rightarrow \sum_{n=1}^{\infty} d_n sin(nx) = 1, \quad u(x,0) = 0 \Rightarrow \sum_{n=1}^{\infty} nc_n sin(nx) = 0 \tag{14.9}$$

so multiplying by $sin(mx)$ and integrating we get (using the orthogonality of the sine functions $\int_0^{\pi} sin(mx)sin(nx) = \frac{\pi}{2}\delta_{mn}$)

$$\int_0^{\pi} sin(mx)dx = d_m \frac{\pi}{2}, \quad \int_0^{\pi} 0dx = mc_m \frac{\pi}{2}. \tag{14.10}$$

Therefore

$$d_m = \frac{2}{m\pi}(1 - con(n\pi)) = \begin{cases} \frac{4}{m\pi} & m\ odd \\ 0 & m\ even \end{cases}; \quad c_m = 0; \tag{14.11}$$

and we get

$$b_n(t) = \begin{cases} \frac{4}{n\pi} cos(nt) & m\ odd, \\ 0 & m\ even; \end{cases} \tag{14.12}$$

and in the end we get the solution

$$u(x,t) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{cos((2n+1)t)sin((2n+1)x)}{(2n+1)}. \tag{14.13}$$

## 14.2 Heat equation

Solve the following problems for the heat equation:

1. $u_t = \Delta u + 1, \quad u(\vec{x}, 0) = 1, \quad t > 0, \ \vec{x} \in \mathbb{R}^5;$
2. $u_t = u_{xx}, \quad u(x,0) = x^2 - x + 1, \ u(0,t) = 1, \ u(2,t) = 3, \quad x \in [0,2], \ t > 0.$

To solve the heat equation we can use, for example, the method of the kernel or the separation of variables. In the first approach we use Fourier transform to write

$$\widehat{(u_t)}(\vec{k}, t) = \frac{\partial}{\partial t}\hat{u}(\vec{k}, t), \quad \widehat{\Delta u}(\vec{k}, t) = -|\vec{k}|^2 \hat{u}(\vec{k}, t), \tag{14.14}$$

to rewrite the $n$-dimensional homogeneous heat equation as

$$\frac{\partial}{\partial t}\hat{u}(\vec{k}, t) = -|\vec{k}|^2 \hat{u}(\vec{k}, t), \tag{14.15}$$

whose solution is $\hat{u}(\vec{k},t) = Ce^{-|\vec{k}|^2 t}$. The initial condition is given by $\hat{u}(\vec{k},0) = \hat{f}(\vec{k})$ from which we get

$$C = \hat{f}(\vec{k}), \tag{14.16}$$

therefore the solution is given by

$$\hat{u}(\vec{k},t) = \hat{f}(\vec{k})e^{-|\vec{k}|^2 t} \Rightarrow u(\vec{x},t) = \widehat{\hat{f}(\vec{k})e^{-|\vec{k}|^2 t}} \tag{14.17}$$

performing the Fourier inversion we get

$$u(\vec{x},t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|\vec{x}-\vec{y}|^2}{4t}} f(y)dy := u^K(\vec{x},t). \tag{14.18}$$

For a non-homogeneous heat equation with non-homogeneous term $g(\vec{x},t)$ with non-homogeneous initial condition we have, using Duhamel principle,

$$\begin{aligned} u(\vec{x},t) &= u^K(\vec{x},t) + u^D(\vec{x},t) = \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|\vec{x}-\vec{y}|^2}{4t}} f(\vec{y})d^n y + \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|\vec{x}-\vec{y}|^2}{4(t-s)}} g(\vec{y},s)d^n y ds. \end{aligned} \tag{14.19}$$

Another useful method to solve homogeneous heat equation is the separation of variables. First of all we need to find a function

$$v(\vec{x},t) = u(\vec{x},t) + (a\vec{x} + b) \tag{14.20}$$

with $a, b$ such that $v(0,t) = 0,\ v(2,t) = 0$. Then we perform a separation of variables $v(\vec{x},t) = X(\vec{x})T(t)$ and inserting in the original equation we end with a system of two ODEs. Taking into account the initial condition we are able to fix the coefficients of the expansion of the function $v(\vec{x},t)$; returning to $u(\vec{x},t)$ we get the solution.

Let us start with the first case. Using the kernel method we have (using gaussian integral with $a = e^{-\frac{x_1^2}{4t}}, b = \frac{1}{4t}, c = 2x_1, d = 0$)

$$\begin{aligned} u^K(\vec{x},t) &= \frac{1}{(4\pi t)^{\frac{5}{2}}} \int_{\mathbb{R}^5} e^{-\frac{|\vec{x}-\vec{y}|^2}{4t}} d^5 y = \\ &= \frac{1}{(4\pi t)^{\frac{5}{2}}} \left( \int_{-\infty}^{+\infty} e^{-\frac{(x_1-y_1)^2}{4t}} dy_1 \right) ... \left( \int_{-\infty}^{+\infty} e^{-\frac{(x_5-y_5)^2}{4t}} dy_5 \right) = \\ &= \frac{1}{(4\pi t)^{\frac{5}{2}}} \left( \sqrt{4t\pi} e^{\frac{x_1^2(16t^2-1)}{4t}} \right) ... \left( \sqrt{4t\pi} e^{\frac{x_5^2(16t^2-1)}{4t}} \right) = \\ &= e^{\frac{|\vec{x}|^2(16t^2-1)}{4t}}, \end{aligned} \tag{14.21}$$

and

$$\begin{aligned} u^D(\vec{x},t) &= \int_0^t \frac{1}{(4\pi(t-s))^{\frac{5}{2}}} \int_{\mathbb{R}^5} e^{-\frac{|\vec{x}-\vec{y}|^2}{4(t-s)}} d^5 y ds = \\ &= \int_0^t e^{\frac{|\vec{x}|^2(16(t-s)^2-1)}{4(t-s)}} ds \end{aligned} \tag{14.22}$$

So the solution is given by

$$u(\vec{x},t) = e^{\frac{|\vec{x}|^2(16t^2-1)}{4t}} + \int_0^t e^{\frac{|\vec{x}|^2(16(t-s)^2-1)}{4(t-s)}} ds. \tag{14.23}$$

In the second case we use the separation of variable method. Let us put

$$v(x,t) = u(x,t) + (ax+b) \tag{14.24}$$

with $a, b$ such that

$$\begin{aligned} v(0,t) = 0 \Rightarrow 0 = u(0,t) + b = 1 + b \Rightarrow b = -1; \\ v(2,t) = 0 \Rightarrow 0 = u(2,t) + 2a - 1 = 2a + 2 \Rightarrow a = -1; \end{aligned} \tag{14.25}$$

therefore

$$v(x,t) = u(x,t) - x - 1. \tag{14.26}$$

We have reduced the problem to $v_t = v_{xx},\ v(x,0) = x^2 - x + 1 - x - 1 = x^2 - 2x,\ v(0,t) = 0,\ v(2,t) = 0,\ x \in [0,2],\ t > 0$. We now put

$$v(x,t) = X(x)T(t) \tag{14.27}$$

and from the boundary conditions we have

$$v(0,t) = X(0)T(t) = 0, \quad v(2,t) = X(2)T(t) = 0 \Rightarrow X(0) = X(2) = 0. \tag{14.28}$$

Interting in the equation we get

$$XT' - X''T = 0 \Rightarrow \frac{X''}{X} = \frac{T'}{T} \tag{14.29}$$

Since $\frac{X''}{X}$ and $\frac{T'}{T}$ are independent functional ratios, the only possibility is that they are equal to the same constant $-\lambda$, we get

$$X'' + \lambda X = 0, \quad T' + \lambda T = 0. \tag{14.30}$$

From the first equation we get

$$X(x) = acos(\sqrt{\lambda}x) + bsin(\sqrt{\lambda}x); \tag{14.31}$$

and using the boundary condition $X(0) = X(2) = 0$ we get

$$X(0) = a = 0, \tag{14.32}$$

so $X(x) = bsin(\sqrt{\lambda}x)$, and

$$X(2) = bsin(2\sqrt{\lambda}) = 0 \Rightarrow \lambda = \left(\frac{n\pi}{2}\right)^2. \tag{14.33}$$

Since there is an $n$ in $\lambda$, this means that we have a set of function $X_n(X)$ that are avaible solutions instead of only one:

$$X_n(x) = b_n sin\left(\frac{n\pi x}{2}\right). \tag{14.34}$$

Knowing $\lambda$ we can solve the equation for $T(t)$ (we will get a set of function labelled by $n$ for the same reason of before), we get

$$T_n(t) = c_n e^{-(\frac{n\pi}{2})^2 t}; \tag{14.35}$$

the solutions is so

$$v(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} b_n c_n sin\left(\frac{n\pi x}{2}\right)e^{-(\frac{n\pi}{2})^2 t} = \sum_{n=1}^{\infty} d_n sin\left(\frac{n\pi x}{2}\right)e^{-(\frac{n\pi}{2})^2 t} \tag{14.36}$$

where $d_n = b_n c_n$. The sum is there because since each $v_n(x,t))X_n(x)T_n(t)$ is a solution, the most general solution will be given by a combination of $v_n(x,t)$ (this is true only because the heat equation is linear). The coefficients $d_n$ can be found using the initial condition

$$v(x,0) = x^2 - 2x \Rightarrow x^2 - 2x = \sum_{n=1}^{\infty} d_n sin\left(\frac{n\pi x}{2}\right) \tag{14.37}$$

from which, using ortogonality of sine function, we get

$$d_m = \int_0^2 (x^2 - 2x) sin\left(\frac{m\pi x}{2}\right) = \begin{cases} 0 & m \ even \\ -\frac{32}{(n\pi^3)} & m \ odd \end{cases}. \tag{14.38}$$

The solution is so

$$v(x,t) = \sum_{n=0}^{\infty} -\frac{32}{((2n+1)\pi^3)} sin\left(\frac{(2n+1)\pi x}{2}\right)e^{-\left(\frac{(2n+1)\pi}{2}\right)^2 t}, \tag{14.39}$$

and returning back to function $u(x,t)$ we find the solution of the original problem

$$u(x,t) = \sum_{n=0}^{\infty} -\frac{32}{((2n+1)\pi^3)} sin\left(\frac{(2n+1)\pi x}{2}\right)e^{-\left(\frac{(2n+1)\pi}{2}\right)^2 t} + x + 1. \tag{14.40}$$