Sapienza
Università di Roma

AdS/CFT extensions: unoriented quiver gauge theories

Faculty of mathematical, physical and natural science
Corso di Laurea Magistrale in Theoretical Physics

Candidate
Federico Manzoni
ID number 1759892

Thesis Advisor
Prof. Fabio Riccioni

Academic Year 2020/2021

Thesis not yet defended

AdS/CFT extensions: unoriented quiver gauge theories
Master's thesis. Sapienza – University of Rome

This thesis has been typeset by LaTeX and the Sapthesis class.

Author's email: federico13.manzoni97@gmail.com

Introduction: the beautiful strangeness of our world

The language of our world, at the microscopic length scale of particles, is provided by Quantum Field Theory; this is the perfect matching between Special Relativity and Quantum Physics. However Quantum Field Theory is something different from the bare union of this two milestones of physics: physicists understood that quantum mechanics is not compatible with the causality principle of Special Relativity and, in fact, they formulated Quantum Field Theory in terms of local operators since these degrees of freedom are more suitable to solve the issues related to causality. Moreover, the development of the theory is based on symmetry principles.

This framework is the basis of the Standard Model of particles physics, which describes three of the four fundamental interactions of Nature: electromagnetism, weak and strong force. In this context, below of the Plank scale, gravity is completely negligible compared with the other forces.

Electromagnetism interaction was described in terms of the Quantum Field Theory framework of the celebrated Quantum ElectroDynamics. It was understood that electrons and photons could be thought, as usual in Quantum Field Theory, as excitations of quantum fields and Quantum ElectroDynamics was formulated by the requirement of the existence of local $U_e(1)$ symmetry from Tomonaga, Schwinger, Feynman and Dyson. Computations were carried out in perturbation theory and the Ultra-Violet divergences arising in loop diagrams were cured by the so-called renormalization procedure. The success of Quantum Electrodynamics was incredible since it reproduces for example, with an amazing accuracy, the magnetic moment of the electron. The generalization to more complicated gauge groups was conceived by Yang and Mills: they argued that gauge symmetry can be non abelian and this idea was at the base of more tricky interactions as weak and strong forces. Moreover, the proposal of the Higgs mechanism, led to the formulation of the celebrated electroweak $SU(2)_L \times U(1)_Y$ theory from Glashow, Salam and Weinberg. The strong force is instead described by Quantum ChromoDynamics that was largely formulated by Gell-Mann, Gross, Wilczek, Politzer and based on the $SU_c(3)$ gauge group algebra. Thus it was that in the mid-seventies the Standard Model was formulated definitely and this model has been verified over the years in the most varied accelerators in the world culminate with the recent observation, in 2012 at the Large Hadron Collider, of a resonance with the characteristics of the Higgs boson; moreover the theoretical predictions of the Standard Model are in great agreement with the experimental data collected by accelerators.

There is an important difference between the electroweak sector and the strong sector: the first is the paradigm of a weakly coupled quantum field theory, in which the observed particles are in one to one correspondence with the fields appearing in the action, whose interactions, treated perturbatively, predict the results of the scattering processes; the second is instead an example of a theory that admits a strongly coupled regime: there is a regime in which the coupling constant of the theory does not allow the perturbation expansion and in which the physical degrees of freedom are not those described at the action level. This is the regime where quarks are confined into hadrons.

The study of strongly coupled regimes is vital to increase our knowledge of the world and to find a place for the new physics that we hope will be found with the new powerful and high luminous accelerators. Indeed, it is commonly accepted that the Standard Model is a low energy effective field theory that requires modifications at higher energies and the consistent introduction of quantum gravity at the Plank scale. Most of this modifications possess an enlarged space-time symmetry algebra, the supersymmetry algebra and, probably, part of this modifications go in the direction of strongly coupled theories. Moreover, Quantum Field Theory and its strongly coupled regimes are not present only in elementary particle physics but they appear in many other fields of physics. For example in Condensed Matter Physics there are some materials that enjoy a superconducting state at high temperature, they are called strange superconductors. The appellative "strange" refers to the fact that they escape to the description through Bardeen, Cooper and Schrieffer like theories. Indeed, the strong interactions between the components of the system make the usual Condensed Matter Physic quasi-particle description of electrons near the Fermi surface to break down and therefore make the description of the mechanism behind Cooper pairing intractable by the usual weakly coupled Quantum Field Theory techniques. Yet another example is given by Quark-Gluon plasma physics, a state of matter where the quarks are not confined and color charges are free. This exotic state of matter is important to study the evolution of the primordial Universe and it can not be studied with weakly coupled methods.

Most of the understanding and of the knowledge that we have about the strong coupling dynamics is due to the so-called dualities. Sometimes a field theory, formulated in terms of dynamical fields ψ_i and an action $S[\psi_i]$, can be equivalent to another field theory formulated in terms of other fields ϕ_i and another action $\tilde{S}[\phi_i]$. The equivalence between the two theories manifests itself in all the physically observable quantities, like partition functions, gauge invariant correlators, various indices, and so on and so forth. Even if the two descriptions may be completely different, the theory that they define is physically equivalent. The real strength of duality lies in the fact that often, while one description is strongly coupled, the other one is weakly coupled, allowing perturbative computations to be done. Since in Quantum Field Theory every field is associated to a particle, it may seem that in describing the theory with two sets of fields we are changing the particles content of the theory. This is not true because the identification of the fields with the physical particles is good only at weak coupling, while in a strongly coupled regime the relation between the fields appearing in the action and the physical spectrum is no longer so trivial. It is the case of hadronic low energy regime of Quantum ChromoDynamics, where the spectrum consists of hadrons while the action is written in terms of quarks and gluons degrees of freedom.

The duality that has probably had the greatest impact is the AdS/CFT correspondence, also dubbed gauge/gravity duality, which was proposed in 1997 by Maldacena and that arises from superstring theory. This duality was a great success and aroused great interest because it provides the best realization of the holographic principle, proposed by 't Hooft and Susskind in the early 90s, and according to which the information of a quantum theory of gravity in D-dimensions is completely captured by the $(D-1)$-dimensional gravity free boundary theory. This is why it is also named holography. Thus the very astonishing feature about AdS/CFT correspondence is that this is a duality between theories with different space-time dimensions and this may seem very strange; however we must remember that in Quantum Field Theory the concept of particles is less fundamental especially in the case of strong coupled regimes. This gives the possibility to dualities relating theories in different space-time dimensions in which one description may not contain particles but different degrees of freedom. Infact Maldacena argued that the $\mathcal{N}=4$ superconformal Yang Mills theory, a four-dimensional superconformal gauge theory with sixteen plus sixteen supercharges, is dual to a Quantum Gravity theory living in a five-dimensional anti de-Sitter space-time. From the fact that on the one hand we have an anti de-Sitter background and on the other hand we have a Conformal Field Theory, the name AdS/CFT emerges.

Gauge/gravity duality provides an extremely powerful tool for understanding non perturbative aspects of Quantum Field Theories. This is because, in the 't Hooft limit of large number N of colors, and when the gauge theory side is at strong coupling, the Quantum Gravity theory side is formulated in terms of perturbative superstrings. But there is even more: when we consider the large N, large 't Hooft coupling and small YM coupling limit in the gauge theory side, the superstring theory is described by its low energy limit, namely a semiclassical supergravity theory in which calculations are more tractable.

Despite this good novelty, the 1997 original AdS/CFT correspondence realizes the duality with a field theory side too unrealistic: too much supersymmetry, conformality, non chirality and integrability. If we want to use holography as a tool to study strongly coupled systems we need more realistic gauge theories; possibilities are to reduce the degree of supersymmetry or to consider non conformal theories.

As we will see later on, the AdS/CFT correspondence can be extended to an infinite class of $\mathcal{N}=1$ supersymmetric gauge theories, whose features are determined by the geometry of certain five-dimensional compact manifolds, called Sasaki-Einstein manifolds, whose six-dimensional cone is a Calabi-Yau one. However, for most of these geometries we lose the control and so we must consider only a subclass of them: the so-called toric varieties. The central points of toric geometry, which makes these manifolds more tractable, are its combinatorial character and the fact that the geometry is completely encoded in a polytope: the toric diagram of the variety. The gauge theory that emerges from these geometries can be studied with specific tools, like quiver diagrams and brane tilings, that will be introduced later on.

A way to break conformality are the so-called orientifold planes. They act like mirrors in the gravity side theory and they have the interesting property to make oriented strings unoriented; this has important consequences on the gauge dual theories since some degrees of freedom are projected out following some precise rules. This leads to possible symplectic and orthogonal gauge groups as well as matter in symmetric and antisymmetric representations. Under orientifold projections the field theory may or may not admit a new superconformal fixed point, eventually reached through an Infra-Red Renormalization Group flow. However, last year an interesting behaviour was found: in this case the orientifold theory flows in the Infra-Red reaching a superconformal point and turns out that, at this point, the R-charges, the superconformal charge and the superconformal index are the same as those of another orientifold theory. This is called third scenario and seems to suggest an Infra-Red duality between two orientifold theories. The main goal of this thesis work is to study how orientifold acts on toric diagram, if its action can be read off directly from toric diagram and what discriminates between third scenario and the standard lore.

My thesis work is structured as follow. In the first part, it is given a review of the general necessary background about supersymmetry, supersymmetric field theory and their non perturbative dynamics (Chapter 1), string theory and T-duality (Chapter 2), the original AdS/CFT correspondence and the general framework for extending the holographic duality to less supersymmetric theories (Chapter 3).
In the Chapter 4 we immerse ourselves in the study of toric manifolds and of the tools necessary to face up to the gauge theories that emerge from these toric manifolds; in Section 4.4 we propose a rewrite of the superconformal central charge in terms of the areas of the triangles that triangulate the toric diagram. The interesting point here is that areas are proportional to the work integral of a suitable vector field that contain information about the Sasaki-Einstein background. In Chapter 5 we introduce orientifolds and the consequent possible scenarios for the field theory. In Section 5.2 we motivate why third scenario seems to be possible only for a certain class of orientifold while in Section 5.3 we give a proposal on how orientifolds act on toric admitting the possibility of a not trivial toric diagram automorphism, which could explain the emergence, at least in same cases, of the third scenario.

Contents

Chapter 1

Supersymmetry

Supersymmetry (SUSY) is a space-time symmetry that associates a boson to each fermion and viceversa; this symmetry was proposed and developed by three different research groups during the seventies: J. L. Gervais and B. Sakita in 1971 [1], Yu. A. Golfand and E. P. Likhtman in 1972 [2], and D. V. Volkov and V. P. Akulov also in 1972 [3]. The first SUSY model was formulated in 1974 by Wess and Zumino [125]. Prior this, the first to propose a symmetry between fermions and bosons was Hironari Miyazawa[1] [4],[5] but the big difference between the two proposals was that the one proposed by Miyazawa was an internal symmetry badly broken and therefore Miyazawa's works was ignored for long time. A symmetry that interchanges bosons and fermions must be a space-time symmetry because, due to this interchanges, the spin of the particles is modified and consequently also its property under rotations. The schematic action of supersymmetry generators on particle state is something like

$$Q_1|boson\rangle = |fermion\rangle,$$
$$Q_2|fermion\rangle = |boson\rangle.$$

From its birth supersymmetry was invoked to cure some fundamental problems of modern physics like gauge coupling unification at the Great Unified Theories (GUT) scale, nature of dark matter and the hierarchy problem in the Standard Model (SM). We now discuss briefly this problems and how supersymmetry can resolve them.

- Gauge couplings unification [6],[7],[8]: The SM gauge couplings follow the Renormalization Group (RG) equations and at the scale of the Z^0 boson mass there is a hierarchy between them $g_{U(1)}(M_{Z^0}) < g_{SU(2)}(M_{Z^0}) < g_{SU(3)}(M_{Z^0})$. Due to the RG equations these couplings vary with the energy scale and hierarchy between them changes drastically; supposing no particles out of SM ones, at the scale $\Lambda_{GUT} \simeq 10^{15} GeV$ they tend to meet. This naturally calls for GUT, where the three interactions are unified in a single one. Usually two possible gauge groups can be used to formulate GUTs: $SU(5)$ and $SO(10)$; these groups must broke spontaneously to reproduce the SM gauge group $SU_c(3) \times SU_L(2) \times U_Y(1)$. One of the problems of GUTs is the fact that the three gauge couplings only approximately meet if we admit no other particles

[1]The symmetry was proposed by Miyazawa in the field of hadronic theory and concerned mesons and barions.

beside SM ones. The solution is to incorporate supersymmetry in GUTs so we have supersymmetric great unified theories and now the three gauge couplings meet exactly;

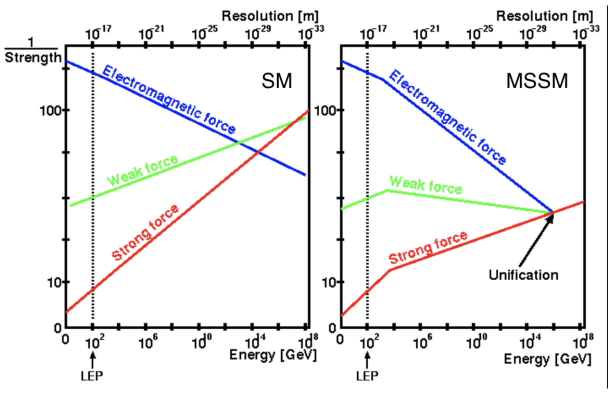

Figure 1.1. *Comparison between the trends of the gauge constants due to the RG flow for the MS and the MSSM. On the left the three constants meet approximately while on the right they meet exactly.*

- Dark matter nature [9],[10]: our universe is made of about 26% by dark matter; this is due to theoretical and experimental reasons[2]. Today we do not know what dark matter is; we only have a set of possible candidate particles and one of them is the neutralino[3], a particle predicted from supersimmetric theories like the Minimal Supersymmetric Standard Model (MSSM);

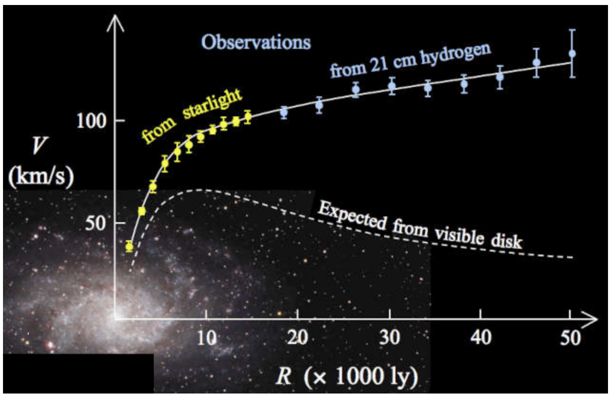

Figure 1.2. *Example of rotation curve of a galaxy. The galaxy's ordinary matter content is unable to explain the observed behaviour; a new type of matter is needed: dark matter.*

[2] About the theoretical point of view we mention the formation of cosmological structures. First structure formed are dark matter halos and then ordinary matter fall in the gravitational potential hole. From experimental point of view, observing the rotational curve of galaxies, we note that the behaviour is in contrast with the matter content of the galaxies, therefore we are forced to consider new matter that have no electromagnetic interaction.

[3] Among other candidates we have the axion particle, associated to the Paccei-Quinn symmetry, and the sterile neutrino, a kind of neutrino that interacts only with gravity. Another one possible solution can be accept a Modified Gravity Theories.

- The hierarchy problem of SM [11]: in the SM the radiative corrections to the Higgs mass are due to Yukawa fermion-antifermion coupling to the Higgs, $-\lambda_f f H \bar{f}$. The one loop correction diverges quadratically and we are forced to introduce a cut-off, Λ_{UV}; the quantum correction is $\Delta M_H^2 = -\frac{|\lambda_f|^2}{8\pi^2}(\Lambda_{UV}^2 + ...)$. The renormalized Higgs mass should be order of the cut-off but this is not what happens. The measured Higgs mass is $M_H \sim 125 GeV$, so a miraculous fine tuning cancellation between the quadratic radiative correction and the bare Higgs mass must happens: this seems to be very unlikely. A possible solution is to consider scalar field at which the Higgs boson couples. The radiative corrections due to this new coupling diverge quadratically as the fermion one but has opposite sign, $\Delta M_H^2 = \frac{2\lambda_s}{16\pi^2}(\Lambda_{UV}^2 + ...)$. Therefore if the new physics is such that each quark and lepton of the SM were accompanied by two complex scalars, having the same Higgs couplings of the quark and the lepton, $\lambda_s = |\lambda_f|^2$, then all quadratically divergent contribution to the quantum corrections would cancel and the Higgs mass would be stabilized at its tree level value. This possibility is made up by supersymmetric extensions of the SM.

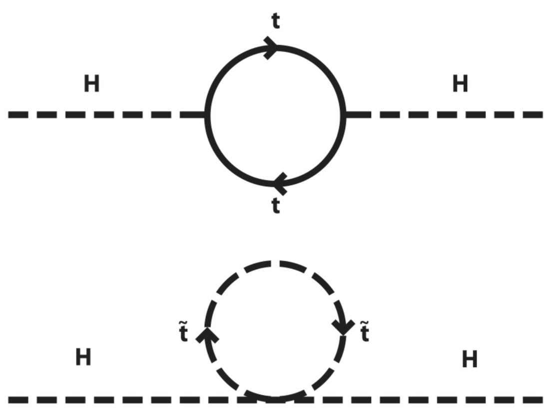

Figure 1.3. *Useful diagrams for the calculation of quantum corrections to the Higgs mass. The main contributions are due to quark and squark loop and they cancel out protecting the Higgs mass.*

Despite supersymmetry resolves a lot of problems, it is important to underline that nowadays there is no experimental evidence of it. Now that we have learned how important is SUSY for modern physics, in the following we will focus on a certain detail on SUSY [12],[13],[14],[17],[18]. First we will study its algebra and its representation and then we will move to the construction of SUSY theories by introducing superfield and superspace formalism. At the end of this chapter we will encounter one of the most important building block for this work: Seiberg duality.

1.1 SUSY algebra and representations

The language of modern particle physics is Quantum Field Theory (QFT) and it is based on Poincaré symmetry and internal symmetry. All particles are classified according to the representation by which they transform under the action of the groups that encode these symmetries. In 1967 Coleman and Mandula [15] asked themselves what were the more general symmetries for the matrix S under some physical and reasonable assumptions[4]: they found that the only possible continuous symmetries of the S matrix are those generated by Poincaré group generators P_μ and $M_{\mu\nu}$ plus some internal symmetry group generator G_a commutating with them

$$[G_a, P_\mu] = [G_a, M_{\mu\nu}] = 0. \tag{1.1}$$

The most general group of symmetry enjoyed by S matrix is the direct product of Poincaré and internal groups: $ISO(1,3) \times G$. One of the crucial assumption of Coleman-Mandula's theorem is that only bosonic generators are allowed, so we can violate it allowing for fermionic generators. The inclusion of both fermionic and bosonic generators involves the use of anticommutator and commutator relations for describing the symmetry algebra; this is exactly the assumption done by Haag, Sohnius and Lopuszaski in 1975 to show that SUSY enlarges the possible S matrix symmetry [16]. Therefore, the most general symmetry group the S matrix enjoys is the product between internal groups and the supersymmetric extension of the Poincaré group. In order to also consider the anticommutators, and so fermionic generators, it is necessary to extend the concept of Lie algebra to the concept of graded Lie algebra[5]. Therefore the supersymmetric extension of Poincaré algebra, also called SuperPoincaré algebra, is a graded Lie algebra of grade $n = 1$

$$L = L_0 \oplus L_1, \tag{1.2}$$

where L_0 is the Poincaré algebra and L_1 contains a set of $2\mathcal{N}$ spinorial generators Q^I_α, $\bar{Q}^{\dot\alpha}_I$ where $I = 1, ..., \mathcal{N}$, $\alpha = 1, 2$ and $\dot\alpha = \dot1, \dot2$. Haag, Lopuszaski and Sohnius showed that the spinorial generators, called also supercharges, must be Weyl fermions trasforming in the $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$ representations of the Lorentz group.

[4]This assumptions are locality, causality, a finite number of particles, etc etc.

[5]Recall that a Lie algebra L is a vector space over some field which enjoys a binary product, $[\cdot,\cdot] : L \times L \to L$, which is antisymmetric, bilinear and satisfies Jacobi identity. A graded Lie algebra L of grade n is a vector space over some field that can be decomposed in direct sum $L = \oplus_{i=0}^{i=n} L_i$ where each L_i is a vector space. Graded Lie algebra has a binary product $[\cdot,\cdot\} : L \times L \to L$ with the following properties:

- $[L_i, L_j\} \in L_{i+j} \text{ mod } n+1$;
- $[L_i, L_j\} = -(-1)^{ij}[L_j, L_i\}$;
- $(-1)^{ik}[L_i, [L_j, L_k\}\} + (-1)^{kj}[L_k, [L_i, L_j\}\} + (-1)^{ji}[L_j, [L_k, L_i\}\} = 0$.

The double parenthesis notation stands for the fact that they can be either commutator or anticommutator.

1.1.1 Supersymmetry algebra

The entire SUSY algebra is

$$\begin{aligned}
&[P_\mu, P_\nu] = 0,\ [M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\sigma\mu},\\
&[M_{\mu\nu}, P_\rho] = -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu,\\
&[P_\mu, G_a] = [M_{\mu\nu}, G_b] = 0,\ [G_a, G_b] = if_{abc}G^c,\\
&[P_\mu, Q^I_\alpha] = [P_\mu, \bar{Q}^{\dot\alpha}_I] = 0,\\
&[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})^\beta_\alpha Q^I_\beta,\ [M_{\mu\nu}, \bar{Q}^{\dot\alpha}_I] = i(\bar\sigma_{\mu\nu})^{\dot\alpha}_{\dot\beta}\bar{Q}^{\dot\beta}_I,\\
&\{Q^I_\alpha, \bar{Q}^{\dot\beta}_J\} = 2(\sigma^\mu)^{\dot\beta}_\alpha P_\mu \delta^I_J,\\
&\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta}Z^{IJ},\ \{\bar{Q}^{\dot\alpha}_I, \bar{Q}^{\dot\beta}_J\} = \epsilon^{\dot\alpha\dot\beta}Z^*_{IJ},\\
&[Q^I_\alpha, G_a] = (g_a)^I_J Q^J_\alpha,\ [\bar{Q}^{\dot\alpha}_I, G_a] = -\bar{Q}^{\dot\alpha}_J (g_a)^J_I,
\end{aligned} \tag{1.3}$$

where $\eta_{\mu\nu}$ is the flat Minkowsky metric, f_{abc} are the structure constants of the internal group, $(\sigma_{\mu\nu})^\beta_\alpha$ and $(\bar\sigma_{\mu\nu})^{\dot\alpha}_{\dot\beta}$ are the Lorentz generators respectively for $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$ Weyl spinors, $(\sigma^\mu)^{\dot\beta}_\alpha$ are the space-time Pauli matrices, $\epsilon_{\alpha\beta}$ and $\epsilon^{\dot\alpha\dot\beta}$ are the $SL(2,\mathbb{C})$ Levi-Civita invariant tensors, Z_{IJ} are the central charges[6] and $(g_a)^I_J$ are the charges of the spinorial generators under the internal symmetry group.
The first two lines of the algebra 1.3 are the Poincaré commutation relations; the third line shows that internal symmetry commutes with Poincaré generators and the Lie algebra of the internal symmetry group; the fourth line shows that traslations are invariant under the action of SUSY generators while the fifth line shows that supersymmetry generators are Weyl spinors; sixth line tells us that two SUSY transformations are equivalent to a traslation[7] and in the seventh line we have the central charges that generate the centre of the SUSY algebra: when the central charges are zero we talk about minimal supersymmetry while when some of the central charges are not zero we have extended supersymmetry. Note that due to the antisymmetry property of the commutators the central charge must satisfy the obvious relation $Z^{IJ} = -Z^{JI}$. The last line shows that the spinorial generators are charged with respect to the internal symmetry group; the largest possible internal symmetry group which can act in a non-trivial way on the SUSY generators is $U(\mathcal{N})_R$ and this is called R-symmetry group. It is important to note that when the central charges are not zero the R-symmetry group reduces to compact symplectic group $USp(\mathcal{N})_R$[8]. The abelian subgroup $U(1)_R$ of the R-symmetry group acts like a phase and its generators are called R-charges. However if the internal symmetry group is a gauge group, the last line of commutator must vanishes. Using SUSY algebra is a simple exercise to show that Q^I_1 and $\bar{Q}^{\dot 2}_I$ rise the z-component of the

[6] A central charge is an operator that commute with all the generator of the algebra.

[7] This means that local supersymmetry automatically incorporates diffeomorphism invariance and so every local SUSY theory is a supergravity theory (SUGRA).

[8] The compact symplectic group contains the $\mathcal{N} \times \mathcal{N}$ matrices belonging to both $U(\mathcal{N})$ and $Sp(\mathcal{N})$.

spin by half unit while Q^I_2 and $\bar{Q}^{\dot{1}}_I$ lower it by the same amount: infact

$$[M_{12}, Q^I_1]|J_3 = 0\rangle \overbrace{=}^{M_{12}=J_3} J_3 Q^I_1 - Q^I_1 J_3 |J_3 = 0\rangle = J_3 Q^I_1 |J_3 = 0\rangle =$$
$$\underbrace{=}_{SUSY\ algebra} i(\sigma_{12})_1^{\beta} Q^I_{\beta}|J_3 = 0\rangle = \frac{1}{2} Q^I_1 |J_3 = 0\rangle \Rightarrow J_3(Q^I_1|J_3 = 0\rangle) = \frac{1}{2}(Q^I_1|J_3 = 0\rangle); \tag{1.4}$$

in the same way it is possible to show the other actions of the supercharge. In a similar trivial manner and using the last line of the 1.3 we find that Q^I_α rise by a unit the R-charge while $\bar{Q}^{\dot{\alpha}}_I$ lower it.

1.1.2 Supersymmetry representations

Now that we have SUSY algebra we are ready to construct supermultiplets, namely irreducible representations of the supersymmetry algebra. As well as in non-supersymmetric theories particles are irreducible representations of Poincaré algebra, in SUSY theories particles are irreducible representations of the SuperPoincaré algebra; this is the reason why we now focus on SUSY representations.
Let us list four important generic properties that any SUSY representation enjoys.

1. It is obvious that spin is not a good quantum number to label representations. This is due to equation 1.4. Particles belonging to the same supermultiplet must have different spin otherwise SUSY would not have a symmetry between bosons and fermions. Beside this, looking at the fourth line of the 1.3, we can note that momentum commutes with supercharges and this implies that the Poincaré casimir P^2 is also a SUSY casimir[9]. To summarize, particles belonging to the same supermultiplet have equal mass[10] but different spin.

2. A supermultiplet contains an equal number of bosonic and fermionic degrees of freedom: $n_b = n_f$. To see this we consider the fermion number operator

$$(-1)^{N_f} = \begin{cases} -1 \ fermionic\ state, \\ +1 \ bosonic\ state. \end{cases} .$$

This operator anticommutes with supercharges and so, fixed a representation $\mathcal{R}$ under which take the trace and a $P_\mu \neq 0$, we have

$$\begin{aligned} 0 &= Tr\big(- (-1)^{N_f} \bar{Q}^{\dot{\beta}}_J Q^I_\alpha + (-1)^{N_f} \bar{Q}^{\dot{\beta}}_J Q^I_\alpha\big) = \\ &= Tr\big(- Q^I_\alpha (-1)^{N_f} \bar{Q}^{\dot{\beta}}_J + (-1)^{N_f} \bar{Q}^{\dot{\beta}}_J Q^I_\alpha\big) \\ &= Tr\big((-1)^{N_f}\{Q^I_\alpha, \bar{Q}^{\dot{\beta}}_J\}\big) = 2(\sigma^\mu)^{\dot{\beta}}_\alpha P_\mu \delta^I_J Tr\big((-1^{N_f})\big) \Rightarrow Tr\big((-1^{N_f})\big) = 0. \end{aligned}$$

[9]The other Poincaré casimir, the Pauli-Lubanski vector, is not anymore a casimir because it is proportional to the spin.

[10]This mass degeneracy is not observed, so if supersymmetry is realized it must be broken at sufficiently low energy.

3. The energy of any state in a SUSY theory is always equal or greater than zero. Using 1.3, fixed an arbitrary state $|\cdot\rangle$, we have[11]

$$2(\sigma^\mu)^{\dot\beta}_\alpha\langle\cdot|P_\mu|\cdot\rangle\delta^I_I = \langle\cdot|\{Q^I_\alpha, \bar{Q}^{\dot\beta}_I\}|\cdot\rangle = \langle\cdot|(Q^I_\alpha(Q^I_\alpha)^\dagger + (Q^I_\alpha)^\dagger Q^I_\alpha)|\cdot\rangle = \\ = |(Q^I_\alpha)^\dagger|\cdot\rangle|^2 + |Q^I_\alpha|\cdot\rangle|^2 \geq 0;$$

taking now the trace over Weyl indexes we prove that $4\langle\cdot|P_0|\cdot\rangle = 4\langle\cdot|H|\cdot\rangle \geq 0$. The spectrum is positive defined, and if we consider the vacuum state we find that

$$4\langle 0|P_0|0\rangle = |(Q^I_\alpha)^\dagger|0\rangle|^2 + |Q^I_\alpha|0\rangle|^2,$$

so supersymmetry is unbroken if and only if the vacuum energy vanishes. If it is not the case, SUSY can not be realized à la Wigner and it is broken.

4. Since SUSY generators commute with the gauge symmetry group, all particle content in the same supermultiplet has to transform in the same way under the gauge group action.

We now construct explicit representation of SUSY algebra. Considering that the mass is a conserved quantity in a supermultiplet we must distinguish between massless and massive representations.

Massive minimal SUSY

We start with massive representations and vanishing central charges, $Z^{IJ} = 0$. We move to the rest frame in which $P_\mu = (m, 0, 0, 0)$ so, looking at the SUSY algebra 1.3,

$$\{Q^I_\alpha, \bar{Q}^{\dot\beta}_J\} = 2m\delta^{\dot\beta}_\alpha\delta^I_J.$$

Defining a set of $2\mathcal{N}$ creation and $2\mathcal{N}$ annihilation operators such as

$$a^I_{1,2} := \frac{1}{\sqrt{2m}}Q^I_{1,2}, \ (a^{\dot 1,\dot 2}_I)^\dagger := \frac{1}{\sqrt{2m}}\bar{Q}^{\dot 1,\dot 2}_I, \tag{1.5}$$

we can act on a vacuum $|m, j\rangle$ defined by mass m, spin j and by the relation $a^I_{1,2}|m, j\rangle = 0$, called Clifford vacuum, to create supermultiplets. Note that the operators 1.5 satisfies the usual Fermi oscillator algebra.

$\mathcal{N} = 1$ massive minimal SUSY

From the general definition 1.5 we have

$$a_{1,2} := \frac{1}{\sqrt{2m}}Q_{1,2}, \ (a^{\dot 1,\dot 2})^\dagger := \frac{1}{\sqrt{2m}}\bar{Q}^{\dot 1,\dot 2}; \tag{1.6}$$

this is a set of 2 creation and 2 annihilation operators and $(a^{\dot 2})^\dagger$ rises the spin while $(a^{\dot 1})^\dagger$ lowers it. Starting from a vacuum with $j = 0$ we can act with $(a^{\dot 2})^\dagger$ and obtain a state with spin $j = \frac{1}{2}$; now we can not act again with the same operator because this

[11] Recall that $(Q^I_\alpha)^\dagger = \bar{Q}^{\dot\beta}_I$.

is a fermion one but we can act with $(a^{\dot{1}})^\dagger$ and obtain a state with $j = 0$. Obviously this state is equivalent to the state created acting previously with $(a^{\dot{1}})^\dagger$ on Clifford vacuum and then act with $(a^{\dot{2}})^\dagger$. A new independent state can be created acting on the Clifford vacuum with the operator $(a^{\dot{1}})^\dagger$ obtaining a state with $j = -\frac{1}{2}$. In conclusion we have a matter supermultiplet

$$|m, j = 0\rangle \Rightarrow \Big(-\frac{1}{2}, 0, 0, \frac{1}{2} \Big), \tag{1.7}$$

with one massive Majorana fermion and one massive complex scalar[12]. In the same manner we can construct a vector multiplet starting from a vacuum with $j = \frac{1}{2}$

$$\Big|m, j = \frac{1}{2}\Big\rangle \Rightarrow \Big(-1, -\frac{1}{2}, -\frac{1}{2}, 0, 0, +\frac{1}{2}, +\frac{1}{2}, 1 \Big), \tag{1.8}$$

so we have a massive vector boson with spin 1, one massive Dirac fermion and one massive real scalar. If we move towards $\mathcal{N} = 2$ massive minimal SUSY, we would immediately see that we would have states with spin greater than one: this theories describe supergravity and we will not care much about that.

Massive extended SUSY

We briefly describe what non-vanishing central charges change. Without loss of generality we consider $\mathcal{N}$ even; a this point is useful a change of basis in the generators space with the aim of putting the matrix of central charges into the form

$$Z^{IJ} = \begin{pmatrix} 0 & Z_1 & \dots & \dots & \dots & \dots & 0 \\ -Z_1 & 0 & \ddots & \dots & \dots & \dots & 0 \\ \vdots & \ddots & 0 & Z_2 & \dots & \dots & 0 \\ \vdots & \vdots & -Z_2 & 0 & \ddots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & Z_{\frac{\mathcal{N}}{2}} \\ 0 & 0 & 0 & 0 & 0 & -Z_{\frac{\mathcal{N}}{2}} & 0 \end{pmatrix}; \tag{1.9}$$

We can now define a set of operators

$$\begin{aligned} &a^1_\alpha := \frac{1}{\sqrt{2}}\Big(Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\Big),\ b^1_\alpha := \frac{1}{\sqrt{2}}\Big(Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\Big); \\ &\dots := \dots,\ \dots := \dots; \\ &a^{\frac{\mathcal{N}}{2}}_\alpha := \frac{1}{\sqrt{2}}\Big(Q^{\mathcal{N}-1}_\alpha + \epsilon_{\alpha\beta}(Q^{\mathcal{N}}_\beta)^\dagger\Big),\ b^{\frac{\mathcal{N}}{2}}_\alpha := \frac{1}{\sqrt{2}}\Big(Q^{\mathcal{N}-1}_\alpha - \epsilon_{\alpha\beta}(Q^{\mathcal{N}}_\beta)^\dagger\Big); \end{aligned} \tag{1.10}$$

and, using the SUSY algebra, we can easily see that the operators 1.10 satisfy the following algebra

$$\begin{aligned} &\{a^r_\alpha, (a^s_\beta)^\dagger\} = (2m + Z_r)\delta_{sr}\delta_{\alpha\beta}; \\ &\{b^r_\alpha, (b^s_\beta)^\dagger\} = (2m - Z_r)\delta_{sr}\delta_{\alpha\beta}; \\ &\{a^r_\alpha, a^s_\beta\} = \{b^r_\alpha, b^s_\beta\} = \{a^r_\alpha, (b^s_\beta)^\dagger\} = \{b^r_\alpha, (a^s_\beta)^\dagger\} = 0; \end{aligned} \tag{1.11}$$

[12] Note that this multiplet is self CPT.

where the indexes r, s run over $1, ..., \frac{\mathcal{N}}{2}$. We have constructed a set of $2\mathcal{N}$ creation and $2\mathcal{N}$ annihilation operators. From the second anticommutator and from the third point of the generic properties list of each SUSY representation, we find the constrain

$$2m \geq |Z_r|, \tag{1.12}$$

this is an important information. We can have three cases:

- the constrain is never saturate $\forall r \Rightarrow$ we have the whole set of $2\mathcal{N}$ creation and $2\mathcal{N}$ annihilation operators. One talk about long multiplets;
- the constrain is saturate for some Z_r for example for $r = k < \frac{\mathcal{N}}{2}$ of them $\Rightarrow$ looking at the algebra 1.11 we see that k of b-type operators become trivially realized. One talk about short multiplets;
- the constrain is saturated from all the $Z_r \Rightarrow$ half of the creation operators trivialized and we call this representations ultra short multiplets.

Multiplets belonging to the second or third point of this list are called $\frac{k}{\mathcal{N}}$-BPS states[13], they are supersymmetry preserving states. This concludes our discussion on massive extended SUSY.

Massless representations

As per the title above, massless representations are not distinguished into minimal and extended. This can be seen from the condition 1.12: if $m = 0$ then all Z_r vanish and therefore massless representations are only minimal.

For a massless particle we can go to the rest frame and so $P_\mu = (E, 0, 0, E)$. In such frame we have

$$\{Q^I_\alpha, \bar{Q}^{\dot{\beta}}_J\} = 2(\sigma^\mu)_\alpha^{\dot{\beta}} P_\mu \delta^I_J = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix} \delta^I_J \Rightarrow \{Q^I_1, \bar{Q}^{\dot{1}}_J\} = 0, \tag{1.13}$$

this anticommutator implies that the supercharges Q^I_1 and $Q^I_{\dot{1}}$ are trivially realized and so we are left with only half of the SUSY generators: this means that massless representations are long as ultra short massive representations. From the non-trivial generators we can define a set of creation and annihilation operators

$$a_I := \frac{1}{\sqrt{4E}} Q^I_2, \;\; a^\dagger_I := \frac{1}{\sqrt{4E}} \bar{Q}^{\dot{2}}_I. \tag{1.14}$$

These operators satisfy a set of anticommutation relations that are easily derived starting from anticommutation relations of the supercharges

$$\{a_I, a^\dagger_J\} = \delta_{IJ}, \;\; \{a_I, a_J\} = \{a^\dagger_I, a^\dagger_J\} = 0. \tag{1.15}$$

This set of $\mathcal{N}$ creation and $\mathcal{N}$ annihilation operators are the basic tools to construct irreducible representations of SUSY algebra. In order to construct representations we need a vacuum state annihilated by all the a_I: the Clifford vacuum. This vacuum

[13] This name is due to the fact that the constrain 1.12 recall the Bogomol'nyi–Prasad–Sommerfield bound. The term $\frac{k}{\mathcal{N}}$ count the number of non broken generators.

has $m = 0$ but carries some helicity λ_0 and will be indicated as $|\lambda_0\rangle$. Acting on this vacuum with the creation operators we can construct all states in the representations. Note, however, that due to antisymmetry in I and J we can count the number of states with a given helicity $\lambda_0 + \frac{k}{2}$ with $k \leq \mathcal{N}$. This turn out to be

$$\#\ of\ states\ with\ helicity\ \lambda_0 + \frac{k}{2} = \binom{\mathcal{N}}{k}, \tag{1.16}$$

so we can count the number of states exactly in the same way as we calculate the coefficients of the Newton's binomial theorem.
Last but not least, we have to mention that, if we want a CPT invariant theory[14], we must to double the multiplet; this is becauce CPT flips helicity.

$\mathcal{N} = 1$ massless SUSY

Starting from a Clifford vacuum with $\lambda_0 = 0$ we obtain a matter multiplet, also called chiral or Wess-Zumino multiplet:

$$|\lambda_0 = 0\rangle \Rightarrow \Big(0, +\frac{1}{2}\Big) \ \oplus \ \Big(-\frac{1}{2}, 0\Big), \tag{1.17}$$

the degrees of freedom are those of one Weyl spinor and one complex scalar. This is the multiplet where matter sits. One can identify the fermions of those multiplets as the SM fermions while the complex scalar are identified with their superpartner: sfermions.
Continuing with a Clifford vacuum $|\lambda_0 = +\frac{1}{2}\rangle$ we obtain the gauge or vector multiplet

$$\Big|\lambda_0 = +\frac{1}{2}\Big\rangle \Rightarrow \Big(+\frac{1}{2}, 1\Big) \ \oplus \ \Big(-1, -\frac{1}{2}\Big), \tag{1.18}$$

we have a massless vector and one Weyl fermion. Those are the multiplets used to describe gauge bosons; their SUSY partners are called gaugini.
Those two kind of multiplets are the only possible if one does not care about gravity[15]; in fact starting with a Clifford vacuum with $\lambda_0 \geq +1$ inevitably we end with spin grater than 1.

$\mathcal{N} = 2$ and $\mathcal{N} = 4$ massless SUSY

For $\mathcal{N} = 2$ again we have only two kind of possible multiplets that not lead to local supersymmetry and hence supergravity, those are hypermultiplets and gauge multiplets:

$$\Big|\lambda_0 = -\frac{1}{2}\Big\rangle \Rightarrow \Big(-\frac{1}{2}, 0, 0, +\frac{1}{2}\Big) \ \oplus \ \Big(+\frac{1}{2}, 0, 0, -\frac{1}{2}\Big), \tag{1.19}$$

and

$$|\lambda_0 = -1\rangle \Rightarrow \Big(-1, -\frac{1}{2}, -\frac{1}{2}, 0\Big) \ \oplus \ \Big(0, +\frac{1}{2}, +\frac{1}{2}, 1\Big). \tag{1.20}$$

[14]Nature violates discrete symmetry such as parity or CP, but is a general features that no theory violates CPT in nature.

[15]For $\lambda_0 = \frac{3}{2}$ we obtain the graviton multiplet containing one spin 2 boson and one Rarita-Schwinger field: the gravitino.

The former contains two complex scalars and two Weyl fermions while the latter has one vector, two Weyl fermions and two complex scalars degrees of freedom.
For theories with $\mathcal{N} = 4$ we have only one kind of supermultiplet ignoring those that give rise to particles with spin grater than 1

$$|\lambda_0 = -1\rangle \Rightarrow \Big(-1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, 0, 0, +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, 1\Big); \quad (1.21)$$

hence we have one vector, four Weyl fermions, and three complex scalars. $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SUSY can be recast in the language of $\mathcal{N} = 1$ SUSY: as an example $\mathcal{N} = 4$ supermultiplet can be thought as three CPT invariant Wess-Zumino multiplets plus one CPT invariant $\mathcal{N} = 1$ gauge multiplet.

We remark that this discussion is valid for $D = 4$ space-time dimensions; however the supersymmetry algebra and its representation can be discussed similarly if the space-time dimension is different. Nevertheless, non gravitational multiplets exist only for $D \leq 10$. In the following we focus on $D = 4$ SUSY.

1.2 SUSY theories

In the previous section we talked about SUSY algebra and SUSY representations. Now it is time to construct supersymmetric theories; to this aim we must study fields representations and this is not a trivial task. We must introduce a new formalism to describe $\mathcal{N} = 1$ theory enlarging usual space-time and admitting Grassmann[16] coordinates; this enlarged space is called superspace and fields defined inside it are called superfields. We will see how powerful this formalism is to build SUSY theories.

1.2.1 Superspace and superfields

In ordinary space-time supersymmetry is not manifest so we have to enlarge our framework. The basic idea is to consider the ordinary x^μ coordinates, associated to the generators P_μ, and adding four Grassmann coordinates, θ^α and $\bar{\theta}_{\dot{\alpha}}$, associated to SUSY generators Q_α and $\bar{Q}^{\dot{\alpha}}$. This is the $\mathcal{N} = 1$ superspace. In the following, only $\mathcal{N} = 1$ superspace formalism will be used and it is more than enough to describe theory with $\mathcal{N} > 1$. The explicit construction of superspace is not so different from how Minkowski space is constructed from ordinary Euclidean space. In that case the Minkowski space is defined as the coset space between Poincaré group and Lorentz group

$$\mathbb{M}^{1,3} = \frac{ISO(1,3)}{SO(1,3)}; \quad (1.22)$$

this definition means that every Poincaré transformation is defined up to Lorentz transformation[17] so every coset class, or equivalently, every point in Minkowski space has a unique representative: a translation. Every translation can be parametrized by a coordinate x^μ thanks to the identification

$$x^\mu \Longleftrightarrow e^{ix^\mu P_\mu}. \quad (1.23)$$

[16] Grassmann variables are anticommutating variables and so are the right choice to deal with spinors.

[17] Two Poincaré transformations are declared equivalent if and only if they differ by a Lorentz transformation.

In a very similar way we can define superspace as the coset space

$$\mathbb{S}^{4|1} = \frac{Osp(4|1)}{SO(1,3)}, \tag{1.24}$$

where $Osp(4|1)$ is the superPoincaré group obtained exponentiating the SUSY algebra[18]. A generic point in this coset space can be identified, in the same way as previously, with the coset representative corresponding to the so-called supertranslation

$$(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \Longleftrightarrow e^{ix^\mu P_\mu} e^{i(\theta_\alpha Q^\alpha + \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}})}. \tag{1.25}$$

So $\mathcal{N} = 1$ superspace is a eighth dimensional space with four Grassmann coordinates and four ordinary coordinates.

Now we have the correct space where we can work with supersymmetry; so we have to introduce the objects that live in this superspace: superfields. The most general superfield, recalling some Grassmann analysis, is of the form

$$\begin{aligned} Y(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) = {} & f(x^\mu) + \theta_\alpha \psi^\alpha(x^\mu) + \bar{\theta}_{\dot{\beta}} \bar{\chi}^{\dot{\beta}}(x^\mu) + \theta^\alpha \theta_\alpha m(x^\mu) + \bar{\theta}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}} n(x^\mu) + \\ & + \theta_\alpha (\sigma^\nu)^{\alpha\dot{\beta}} \bar{\theta}_{\dot{\beta}} v_\nu(x^\mu) + \theta^\alpha \theta_\alpha \bar{\theta}_{\dot{\beta}} \bar{\lambda}^{\dot{\beta}}(x^\mu) + \bar{\theta}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}} \theta_\alpha \rho^\alpha(x^\mu) + \\ & + \theta^\alpha \theta_\alpha \bar{\theta}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}} d(x^\mu), \end{aligned} \tag{1.26}$$

where $f(x^\mu), m(x^\mu), n(x^\mu), d(x^\mu)$ are scalar fields, $\psi^\alpha(x^\mu), \rho^\alpha(x^\mu)$ are left Weyl spinor fields , $\bar{\chi}^{\dot{\beta}}(x^\mu), \bar{\lambda}^{\dot{\beta}}(x^\mu)$ are right Weyl spinor fields and $v_\nu(x^\mu)$ is a vector field. In the end a superfield is nothing but a collection of ordinary fields. We refer to F-term for fields that are multiplied by $\theta^\alpha\theta_\alpha \equiv \theta^2$ or by $\bar{\theta}_{\dot{\beta}}\bar{\theta}^{\dot{\beta}} \equiv \bar{\theta}^2$ and we refer to D-term for those ones multiplied by $\theta^\alpha\theta_\alpha\bar{\theta}_{\dot{\beta}}\bar{\theta}^{\dot{\beta}} \equiv \theta^2\bar{\theta}^2$. In the following, when not essential, we will suppress the spinor indexes structure and the coordinate dependence of fields and superfields.

At this point the question is: how does a superfield transform under SUSY trasformation? To find the answer we need to realize supersymmetry generators as differential operators, exactly in the same way it happens for the translation generator. The realization of SUSY generators as differential operators is given by (see appendix A for complete calculation)

$$\mathcal{Q}_\alpha = -i\partial_\alpha - (\sigma^\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu, \;\; \bar{\mathcal{Q}}_{\dot{\beta}} = +i\bar{\partial}_{\dot{\beta}} + \theta^\alpha (\sigma^\mu)_{\alpha\dot{\beta}} \partial_\mu; \tag{1.27}$$

looking at the results of appendix A we note also that the supersymmetric variation of a superfield is represented by

$$\delta_{\zeta,\bar{\zeta}} Y(x,\theta,\bar{\theta}) = (i\zeta\mathcal{Q} + i\bar{\zeta}\bar{\mathcal{Q}}) Y(x,\theta,\bar{\theta}). \tag{1.28}$$

The superfield 1.26 has too many fields components to be an irreducible representation of SUSY algebra, so we can impose some constrains to reduce the field content of a superfield. In the following we will construct some important superfields which we will use often.

[18]SUSY algebra contains anticommutator, so we need some work to recast SUSY algebra using only commutator. Then we have to exponentiate this algebra to obtain superPoincaré group.

Chiral and antichiral superfield

We define the two supercovariant derivatives

$$D_\alpha := \partial_\alpha + i(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu,\ \ \bar D_{\dot\beta} := \bar\partial_{\dot\beta} + i\theta^\alpha(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu, \tag{1.29}$$

these two objects anticommute with the SUSY generators and therefore a constrain build up with the operators 1.29 is a supersymmetric invariant constrain. A chiral superfield Φ is a superfield such as

$$\bar D_{\dot\beta}\Phi = 0, \tag{1.30}$$

in the same way we can impose

$$D_\alpha\bar\Phi = 0 \tag{1.31}$$

and this is an antichiral field. At this point is useful a change of variable in the superspace: $y^\mu = x^\mu + i\theta^\alpha(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}$ and $\bar y^\mu = x^\mu - i\theta^\alpha(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}$; implementing constrains 1.30, 1.31 in the just defined coordinates, we get

$$\begin{aligned}\Phi(y,\theta) &= \phi(y) + \sqrt{2}\theta\psi(y) - \theta^2F(y) = \\ &= \phi(x) + \sqrt{2}\theta\psi(x) + i\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \theta^2F(x) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar\theta - \frac{1}{4}\theta^2\bar\theta^2\Box\phi(x)\end{aligned} \tag{1.32}$$

and

$$\begin{aligned}\bar\Phi(\bar y,\bar\theta) &= \bar\phi(\bar y) + \sqrt{2}\bar\theta\bar\psi(\bar y) - \bar\theta^2F(\bar y) = \\ &= \bar\phi(x) + \sqrt{2}\bar\theta\bar\psi(x) - i\theta\sigma^\mu\bar\theta\partial_\mu\bar\phi(x) - \bar\theta^2\bar F(x) + \frac{i}{\sqrt{2}}\bar\theta^2\theta\partial_\mu\bar\psi(x)\sigma^\mu - \frac{1}{4}\theta^2\bar\theta^2\Box\bar\phi(x).\end{aligned} \tag{1.33}$$

We see that they contain a complex scalar, a Weyl spinor, and another complex scalar, which is an auxiliary field. The latter appears here naturally due to the general definition of a superfield, and whatever action one constructs, it will always be not dynamical. The content of a chiral superfield is then the one of a $\mathcal{N} = 1$ chiral supermultiplet, justifying its name, plus a not dynamical auxiliary field. Sums and products of chiral superfields are chiral superfields; the same holds for antichiral fields. More generally, a crucial fact is that every holomorphic function $W(\Phi)$ of a chiral superfield is a chiral superfield[19] and every anti holomorphic function $\bar W(\bar\Phi)$ of an antichiral superfield is an antichiral superfield.

Vector superfield

We can define a new type of superfield imposing reality condition on the generic superfield 1.26. Vector or real superfields are defined by

$$V = \bar V, \tag{1.34}$$

and the most general field satisfying this condition contains, in addition to the vector supermultiplet fields content, an auxiliary field D plus three real scalars, C, M, N

[19] It is easy to see: $\bar D_{\dot\beta}W(\Phi) = \frac{\partial W}{\partial\Phi}\bar D_{\dot\beta}\Phi + \frac{\partial W}{\partial\bar\Phi}\bar D_{\dot\beta}\bar\Phi = \frac{\partial W}{\partial\Phi}\bar D_{\dot\beta}\Phi = 0.$

and one Weyl spinor χ. Nevertheless, there is a way to reduce the degrees of freedom: SUSY gauge transformation. Note, previously, that the field $\Phi + \bar{\Phi}$ satisfies the reality condition and hence is a real superfield; moreover, it can be shown that the transformation $V \to V + \Phi + \bar{\Phi}$ acts on the vector degree of freedom exactly like an abelian gauge transformation. Choosing properly the field Φ and so a gauge transformation we can put to zero the fields C, M, N, χ and recast the real superfield, in the so-called Wess-Zumino gauge[20],

$$V_{WZ}(x,\theta,\bar{\theta}) = \theta\sigma^{\mu}\bar{\theta}A_{\mu}(x) + i\theta^2\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar{\theta}^2 D(x). \tag{1.35}$$

In the Wess-Zumino gauge, the degrees of freedom are those of $\mathcal{N} = 1$ vector supermultuplet plus a not dynamical auxiliary field. The non abelian generalization is conceptually quite straightforward but more complicated from the point of view of calculations: V and Φ become matrices, $V = V_a T^a$ and $\Phi = \Phi_a T^a$ where T^a are the gauge group generators, and the gauge transformation can be written as $e^V \to e^{i\Lambda}e^V e^{-i\Lambda}$ with the identification $\Phi = -i\Lambda$. The important point is that is still possible recast the vector superfield in the Wess-Zumino gauge.

1.2.2 Supersymmetric theories

We are now ready to construct SUSY theories: we have the right formalism (superspace) and the right building blocks (superfields).
The general philosophy is the following: the integral in superspace of any arbitrary superfield $Y(x,\theta,\bar{\theta})$ is a supersymmetric invariant quantity, that is

$$\int d^4x d^2\theta d^2\bar{\theta} Y(x,\theta,\bar{\theta}) \tag{1.36}$$

is manifestly SUSY invariant[21]. This is easily demonstrated: using 1.28 and 1.27 we have

$$\delta_{\zeta,\bar{\zeta}}Y(x,\theta,\bar{\theta}) = [\zeta^{\alpha}\partial_{\alpha} + \zeta_{\dot{\beta}}\partial^{\dot{\beta}} + \partial_{\mu}(-i\zeta^{\alpha}(\sigma^{\mu})_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}} + i\theta^{\alpha}(\sigma^{\mu})_{\alpha\dot{\beta}}\bar{\zeta}^{\dot{\beta}})]Y(x,\theta,\bar{\theta}), \tag{1.37}$$

on the other hand thanks to translationally invariance of the integral we get

$$\begin{aligned} \delta_{\zeta,\bar{\zeta}}\int d^4x d^2\theta d^2\bar{\theta} Y(x,\theta,\bar{\theta}) &= \int d^4x d^2\theta d^2\bar{\theta}\delta_{\zeta,\bar{\zeta}}Y(x,\theta,\bar{\theta}) = \\ = \int d^4x d^2\theta d^2\bar{\theta}[\zeta^{\alpha}\partial_{\alpha} + \zeta_{\dot{\beta}}\partial^{\dot{\beta}} &+ \partial_{\mu}(-i\zeta^{\alpha}(\sigma^{\mu})_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}} + i\theta^{\alpha}(\sigma^{\mu})_{\alpha\dot{\beta}}\bar{\zeta}^{\dot{\beta}})]Y(x,\theta,\bar{\theta}) = 0, \end{aligned} \tag{1.38}$$

because the first two terms return zero for the property of Berezin's integral while the last term is a total derivative.
Supersymmetric invariant actions are constructed integrating a suitably defined superfield; obviously this superfield cannot be generic but it should have the right structure to give rise, upon integration on Grassmann variables, to a lagrangian density[22].

[20]The Wess-Zumino gauge is so defined as the gauge where $C = M = N = \chi = 0$ but no restrictions on A_{μ}, hence we have still the freedom to perform ordinary gauge transformations on the vector field.

[21]We have defined $d^2\theta := \frac{1}{2}d\theta^1 d\theta^2$ and $d^2\bar{\theta} := \frac{1}{2}d\theta^{\dot{1}}d\theta^{\dot{2}}$.

[22]The lagrangian density must have the following properties: real, dimension four and transforming as a scalar under Poincaré transformations.

$\mathcal{N}=1$ matter actions

We know that matter is accommodated into chiral supermultiples; so, to construct matter actions, we have to consider a generic superfield function of a set of Φ and $\bar{\Phi}$, indicated with $K(\Phi^i,\bar{\Phi}_i)$. Consider the integral

$$\int d^2\theta d^2\bar{\theta} K(\Phi^i,\bar{\Phi}_i), \tag{1.39}$$

note that the only contribution to the above integral is the D-term. In order for 1.39 to be an object that describes a supersymmetric lagrangian density, the function $K(\Phi^i,\bar{\Phi}_i)$ must satisfy some properties: first of all it should be a superfield, this ensures supersymmetry invariance; secondly, it should be a scalar and real function; third, $[K(\Phi^i,\bar{\Phi}_i)]=2$ since then its $\theta^2\bar{\theta}^2$ components will have the right dimension for a lagrangian density (four); finally, it should not be function of $\bar{D}_{\dot{\beta}}\Phi$ and $D_\alpha\bar{\Phi}$ because supercovariant derivatives would provide $\theta^2\bar{\theta}^2$ terms contributions giving an higher derivative theory which can not be accepted in a local QFT. The most general expression compatible with all this requests is

$$K(\Phi^i,\bar{\Phi}_i)=\sum_{m,n=1}^{\infty} c_{mn}(\bar{\Phi}_i)^m(\Phi^i)^n, \tag{1.40}$$

where $c_{mn}=c^*_{nm}$. This is called Kähler potential; this name is due to the fact that the fields ϕ^i and $\bar{\phi}_i$ can be used as complex coordinates for a complex manifold with metric defined by the relation

$$K^i_j=\frac{\partial^2 K(\phi^i,\bar{\phi}_i)}{\partial\phi^i\partial\bar{\phi}_j}; \tag{1.41}$$

this is an example of Kähler manifold[23].
Note that all the c_{mn} have mass dimension; the only one that is dimensionless is c_{11}. If we want a renormalizable theory we have to consider only[24]

$$K(\Phi^i,\bar{\Phi}_i)=\bar{\Phi}_i\Phi^i. \tag{1.42}$$

There is yet another possibility to construct SUSY invariant lagrangians: the superpotential. Consider an holomorphic function of a chiral field $W(\Phi^i)$ and integrate it in half superspace $\int d^2\theta W(\Phi^i)\neq 0$; since lagrangian density must be real we have to add the hermitian conjugate[25]

$$\mathcal{L}_{int}=\int d^2\theta W(\Phi^i)+\int d^2\bar{\theta}\bar{W}(\bar{\Phi}_i) \tag{1.43}$$

[23] This is the first example where we see the intersection between supersymmetric field theories and complex geometry. A deep treatment about complex and Kähler geometry will be done in the paragraph 3.4 of Chapter 3.

[24] Remember that a theory is renormalizable if and only if all its parameters have dimension greater or equal than one.

[25] Note that this lagrangian density seems not to satisfy the general lines described at the beginning of the paragraph. The tricky resolution of this problem is due to the fact that any half superspace integral of a chiral field can be recasted as an integral over the full superspace.

where $W(\Phi)$ is called superpotential. To make sure that the lagrangian density has dimension four, is necessary that the superpotential has dimension three; so it must have the form

$$W(\Phi) = \sum_{n=1}^{\infty} a_n (\Phi^i)^n. \tag{1.44}$$

From a counting dimensional analysis we can easily note that a renormalizable theory must have a superpotential at maximum of $n = 3$.
R-symmetry provides an additional restriction for both Kähler potential and superpotential: infact it turns out that the superpotential must have charge 2 while the Kähler potential charge 0 under R-symmetry.
To summarize, the most general SUSY invariant matter lagrangian density is

$$\mathcal{L}_{matter} = \int d^2\theta d^2\bar{\theta}^2 K(\Phi^i, \bar{\Phi}_i) + \int d^2\theta W(\Phi^i) + \int d^2\bar{\theta}\bar{W}(\bar{\Phi}_i), \tag{1.45}$$

where $K(\Phi^i, \bar{\Phi}_i)$ is a real scalar function with dimension two, called the Kähler potential, while $W(\Phi^i)$ is a holomorphic function with dimension three called superpotential; $\bar{W}(\bar{\Phi}_i)$ is its hermitian conjugate. If we want renormalizable theories we have to truncate the Kähler potential and the superpotential series as

$$K(\Phi^i, \bar{\Phi}_i) = \Phi^i \bar{\Phi}_i, \;\; W(\Phi^i) = \sum_{n=1}^{3} a_n (\Phi^i)^n. \tag{1.46}$$

We now give an explicit example of renormalizable SUSY theory; consider lagrangian 1.45 with a single chiral field Φ; in the first integral only the D-terms of the product $\Phi\bar{\Phi}$ will contribute, while in the second two, only the F-terms of the superpotential will contribute. We expand the superpotential in powers of θ as

$$W(\Phi) = W(\phi) + \sqrt{2}\frac{\partial W}{\partial \phi}\bigg|_{\Phi=\phi} \theta\psi - \theta^2\left(\frac{\partial W}{\partial \phi}\bigg|_{\Phi=\phi} F + \frac{1}{2}\frac{\partial^2 W}{\partial\phi\partial\phi}\bigg|_{\Phi=\phi}\psi\psi\right); \tag{1.47}$$

a similar expansion holds for $\bar{W}(\bar{\Phi})$ and inserting all in 1.45 we get

$$\begin{aligned}\mathcal{L}_{matter} = &\overbrace{\partial_\mu\bar{\phi}\partial^\mu\phi + \frac{i}{2}(\partial_\mu\psi\sigma^\mu\bar{\psi} - \psi\sigma^\mu\partial_\mu\bar{\psi}) + \bar{F}F}^{from\ D-terms} + \\ &\underbrace{-\frac{\partial W}{\partial\phi}F - \frac{1}{2}\frac{\partial^2 W}{\partial\phi\partial\phi}\psi\psi - \frac{\partial\bar{W}}{\partial\bar{\phi}}\bar{F} - \frac{1}{2}\frac{\partial^2\bar{W}}{\partial\bar{\phi}\partial\bar{\phi}}\bar{\psi}\bar{\psi}}_{from\ F-terms};\end{aligned} \tag{1.48}$$

it is now obvious why the fields F and $\bar{F}$ are not dynamical: there is no propagation term for the F field and its conjugate. We can integrate out these auxiliary fields using their equation of motion $\bar{F} = \frac{\partial W}{\partial \phi}$ and $F = \frac{\partial \bar{W}}{\partial \bar{\phi}}$ and 1.48 becomes

$$\mathcal{L}_{matter}^{(on-shell)} = \partial_\mu\bar{\phi}\partial^\mu\phi + \frac{i}{2}(\partial_\mu\psi\sigma^\mu\bar{\psi} - \psi\sigma^\mu\partial_\mu\bar{\psi}) - \left|\frac{\partial W}{\partial\phi}\right|^2 - \frac{1}{2}\frac{\partial^2 W}{\partial\phi\partial\phi}\psi\psi - \frac{1}{2}\frac{\partial^2\bar{W}}{\partial\bar{\phi}\partial\bar{\phi}}\bar{\psi}\bar{\psi}. \tag{1.49}$$

If we choose $W(\Phi) = \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3$ we talk about the so-called Wess-Zumino interacting model (WZIM). Developing the cubic term of the superpotential using 1.49 we get an interaction lagrangian density of the form $\mathcal{L}_{\Phi^3} = -g^2\phi^4 - g\phi\psi\psi - g\bar{\phi}\bar{\psi}\bar{\psi}$. This contains quartic self interaction of the scalar field and Yukawa couplings. We see that the coefficient of the quartic self-interaction of the scalar field is related to the Yukawa couplings of the scalar and fermion fields. This implies that the one loop corrections to the scalar propagator due to these interaction terms are both proportional to g^2 and exactly cancel out each other[26]. This property does not hold just at one loop: infact the superpotential of the WZIM turns out to be exact at tree level. This will give us the opportunity to illustrate a general property of supersymmetry, that is how holomorphy in the couplings provides a simple derivation of very powerful non renormalization theorems. The idea of this so-called spurion method in supersymmetric theories is to promote any parameter in the Lagrangian to be the vacuum expectation value of a superfield. In particular, if we focus on the superpotential term in the lagrangian, each coupling can be thought as the bottom component vacuum expectation value of a chiral superfield; the latter is assumed very heavy and thus frozen at its vacuum expectation value. The theory is viewed as an effective theory of a parent UV theory where these heavy fields have been integrated out, so that only their vacuum expectation values remain in the lagrangian and can be treated as spurion fields. Let us apply this method at the WZIM superpotential. The tree level superpotential is

$$W_{tree}(\Phi) = \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3 \tag{1.50}$$

and we ask ourselves what is the form of the effective superpotential after quantum corrections. We promote m and g to spurionic chiral superfields. This allows us to introduce a spurious $U(1)_f$ flavor symmetry and a spurious $U(1)_R$ R-symmetry. For the fields in 1.50 we take the following charges under the two symmetries:

	$U(1)_f$	$U(1)_R$
Φ	1	1
m	-2	0
g	-3	-1

so that the superpotential has R-charge 2 and flavor charge 0. These symmetries are spurious since they are spontaneously broken once the spurion superfields m and g acquire a non vanishing bottom component vacuum expectation value.
After quantum corrections, the superpotential should be holomorphic in Φ, m, g and must still have R-charge 2 and flavor charge 0. The most general form satisfying these conditions is

$$W_{eff} = m\Phi^2 h\left(\frac{g\Phi}{m}\right) = \sum_{n=-\infty}^{+\infty} c_n g^n m^{1-n}\Phi^{2+n}, \tag{1.51}$$

and to reproduce the tree level superpotential we must have $h_{tree} = \frac{1}{2} + \frac{1}{3}\frac{g\Phi}{m}$. The form of the function $h(\frac{g\Phi}{m})$ can be fixed in the following way: consider the limit

[26] Recall that fermion loop comes with a minus sign compared to boson loop.

$g \to 0$ and, for smoothness, we must have $n \geq 0$ and, in order to be in agree with the tree level superpotential, we have $c_0 = \frac{1}{2}$ and $c_1 = \frac{1}{3}$; moreover, always for smoothness, the limit $m \to 0$ fixes $n \leq 1$. In conclusion $n = 0, 1$:

$$W_{eff} = c_0 m\Phi^2 + c_1 g\Phi^3 = \frac{1}{2}\Phi^2 + \frac{1}{3}g\Phi^3 = W_{tree}. \tag{1.52}$$

The superpotential of the WZIM does not receive quantum corrections and so is exact. We are thus discovering a general feature of supersymmetric theories: combining holomorphy of the superpotential with the spurion method and with smoothness requirements in various weak coupling limits, allows to strongly constrain the effective superpotential terms that are generated by quantum corrections.

$\mathcal{N} = 1$ pure super Yang-Mills action

In the paragraph on vector superfield we have introduced the non abelian gauge transformation

$$e^V \to e^{i\bar{\Lambda}} e^V e^{-i\Lambda}; \tag{1.53}$$

moreover we know that $V = V_a T^a$ and $\Lambda = \Lambda_a T^a$ where T^a with $a = 1, ..., dim(G)$ are the group generator of the gauge group G. Now we want to find an invariant action which can be interpreted as a generalization to supersymmetric theories of the Yang-Mills theory: a super Yang-Mills theory (SYM). In the following is always understood the Wess-Zumino gauge for the vector superfield. Defining

$$W_\alpha := -\frac{1}{4}\bar{D}_{\dot{\beta}}\bar{D}^{\dot{\beta}}(e^{-V} D_\alpha e^V),\ \bar{W}_{\dot{\beta}} := -\frac{1}{4}\bar{D}^\alpha D_\alpha (e^V \bar{D}_{\dot{\beta}} e^{-V}), \tag{1.54}$$

we can note that these two quantities transform covariantly under the non abelian gauge transformation 1.53; for example for W_α we have

$$\begin{aligned} &W_\alpha \to -\frac{1}{4}\bar{D}_{\dot{\beta}}\bar{D}^{\dot{\beta}}[(e^{i\Lambda}e^{-V}e^{-i\bar{\Lambda}})D_\alpha(e^{i\bar{\Lambda}}e^V e^{-i\Lambda})] = \\ &= -\frac{1}{4}\bar{D}_{\dot{\beta}}\bar{D}^{\dot{\beta}}[(e^{i\Lambda}e^{-V}e^{-i\bar{\Lambda}})(\underbrace{(D_\alpha e^{i\bar{\Lambda}})}_{=0} e^V e^{-i\Lambda} + e^{i\bar{\Lambda}}(D_\alpha e^V)e^{-i\Lambda} + e^{i\bar{\Lambda}}e^V(D_\alpha e^{-i\Lambda}))] = \\ &= -\frac{1}{4}e^{i\Lambda}[\bar{D}_{\dot{\beta}}\bar{D}^{\dot{\beta}}(e^{-V}(D_\alpha e^V))e^{-i\Lambda} + \underbrace{\bar{D}_{\dot{\beta}}\bar{D}^{\dot{\beta}}(D_\alpha e^{-i\Lambda})}_{=0}] = e^{i\Lambda}W_\alpha e^{-i\Lambda}. \end{aligned} \tag{1.55}$$

We now can expand the superfield W_α in its ordinary field content. This is done by noting that in the Wess-Zumino gauge the vector superfield must satisfy the relation $V^n = 0$ if $n \geq 3$:

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha D + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta(\sigma^\mu D_\mu \bar{\lambda})_\alpha, \tag{1.56}$$

where $D_\mu = \partial_\mu - igA_\mu$ is the covariant derivative and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ is the field strength of the gauge field.
An invariant gauge lagrangian can be built up taking the trace over the gauge group

indexes

$$\begin{aligned}\mathcal{L}_{SYM} =& \frac{\tau}{16\pi i}\int d^2\theta Tr(W^\alpha W_\alpha) + h.c. = \\ &= Tr\Big[-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \frac{i}{g^2}\lambda^\alpha(\sigma^\mu)_{\alpha\dot\beta}D_\mu\bar\lambda^{\dot\beta} + \frac{1}{2g^2}D^2\Big] + \frac{\theta_{YM}}{32\pi^2}Tr(F_{\mu\nu}\tilde F^{\mu\nu}),\end{aligned} \tag{1.57}$$

where $\tilde F^{\mu\nu}$ is the Hodge dual of the field strength and $\tau = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2}$ is the complexified gauge coupling: g^2 is the gauge coupling while θ_{YM} is the Yang-Mills theta term[27]. Note that the gauge coupling appears only in the holomorphic parameter τ and so is called the holomorphic gauge coupling but it is not the physical one because the kinetic term of the gauge field is not canonically normalized. To go to canonical basis we would rescale the fields $(A_\mu, \lambda_\alpha, D) \to g(A_\mu, \lambda_\alpha, D)$ and this rescaling, which seems harmless, hides some pitfalls: infact turns out that the relation between the holomorphic gauge coupling, g_h, and the physical one, g_p, is not analytical. Using the partition function formalism one can derive the exact relation

$$\frac{1}{g_p^2} = Re\Big(\frac{1}{g_h^2}\Big) - \frac{2T(adj)}{8\pi^2}log(g_p), \tag{1.58}$$

where $T(adj)$ is the Dynkin index for the adjoint representation[28].
The next point is to couple matter with SYM.

$\mathcal{N} = 1$ gauge matter actions

We can now put together what we have learned in the previous two paragraphs to construct gauge matter actions. First of all, note that, under the gauge transformation we expect that $\Phi \to e^{i\Lambda}\Phi$, but the Kähler potential for a renormalizable theory, 1.42, would not be gauge invariant:

$$\bar\Phi\Phi \to \bar\Phi e^{-i\bar\Lambda}e^{i\Lambda}\Phi \neq \bar\Phi\Phi. \tag{1.59}$$

The correct expression is simply understood remembering transformation 1.53,

$$K(e^V\Phi, \bar\Phi) = \bar\Phi e^V\Phi \tag{1.60}$$

and this is gauge invariant at sight[29]. The SUSY gauge matter lagrangian density for one chiral superfield is

$$\begin{aligned}\mathcal{L} =& \mathcal{L}_{SYM} + \mathcal{L}_{matter} + \mathcal{L}_{FI} = \\ &= \int d^2\theta d^2\bar\theta^2 K(e^V\Phi, \bar\Phi) + \int d^2\theta W(\Phi) + \int d^2\bar\theta \bar W(\bar\Phi) + \\ &+ \Big(\frac{\tau}{16\pi i}\int d^2\theta Tr(W^\alpha W_\alpha) + h.c.\Big) + \sum_A \xi_A \int d^2\theta d^2\bar\theta V^A,\end{aligned} \tag{1.61}$$

[27]This term implies periodic physics $\theta_{YM} \to \theta_{YM} + 2\pi$ and CP symmetry break. It is a hard problem that needs non perturbative tools but contains very interesting physics like axions.

[28]Dynkin indexes are definend by $Tr[G_a^{(R)}G_b^{(R)}] = T(R)\delta_{ab}$, R is a generic representation.

[29]If we are interested in non renormalizable theory the correct expression would be $K(e^V\Phi, \bar\Phi) = \sum_{m,n=1}^\infty c_{mn}(\bar\Phi)^m(e^V\Phi)^n$.

the last term, known as Fayet Iliopulos term, is here because the gauge group can be not semi simple and hence contains $A = 1, ..., n$ $U(1)$ abelian subgroups. The entire writing of the 1.61 is very long but no so interesting for us in this moment. What we are interested in, is the fact that 1.61 admits a non trivial on-shell scalar potential

$$V(\phi,\bar{\phi}) = \frac{\partial W}{\partial \phi^i}\frac{\partial \bar{W}}{\partial \bar{\phi}_i} + \frac{g^2}{2}\sum_a |\bar{\phi}_i (T^a)^i_j \phi^j + \xi^a|^2 \underbrace{=}_{on-shell} \bar{F}F + \frac{1}{2}D^2; \tag{1.62}$$

supersymmetric vacua are described by its zeros: a sets of vacuum expectation values of the scalar fields $\langle\phi_i\rangle$. These sets must satisfy the equations $\bar{F} = 0$ and $D = 0$, the so-called F-flat and D-flat directions. Supersymmetric vacua exist in the intersections of these flat directions. Generally there is a huge number of such points, and each of them defines a physical theory; however, we note that these theories are not all inequivalent due to gauge redundant symmetry. Different zeros, related by a gauge transformation, have to be identified. The whole space where they intersect up to gauge transformations is called classical moduli space

$$\mathcal{M}_{classical} := \{\langle\phi_i\rangle | D = 0, \bar{F} = 0\}/gauge\ transformations. \tag{1.63}$$

This is called classical because describes the SUSY vacua at the classical level; we can expect that quantum corrections completely lift the classical flat directions. However thanks to powerful non renormalization theorems, classical moduli space does not have any modification in perturbation theory. What can happens is that there are non perturbative corrections and they can lift or deform classical moduli space.

To conclude this paragraph we want to give a brief alternative description of classical moduli space referring to [19]. The crucial point in this analysis is the observation that the SUSY lagrangian density is invariant under the action of the complexified gauge group[30] $G^{\mathbb{C}}$. From this point of view the D-flatness conditions are a Wess–Zumino gauge artifact. This extended gauge invariance is lost in the Wess-Zumino gauge but it is possible to make it manifest choosing a different gauge. Choosing this gauge is possible to show that every constant matter field configuration that extremizes the superpotential is $G^{\mathbb{C}}$ gauge equivalent to a unique classical vacuum. The consequence is that we do not need to find the solution of the D-term equations, but simply quotient out $G^{\mathbb{C}}$. Specifically if a superpotential is present, the classical moduli space can be re-written as

$$\mathcal{M}_{classical} = \mathcal{F}/G^{\mathbb{C}}, \tag{1.64}$$

where $\mathcal{F}$ is the algebraic variety defined by F-term equations. In [19] is also shown that when a superpotential is present, the classical moduli space is a variety defined by imposing additional relations, due to F-terms equations, on the parameterizing gauge invariant holomorphic polynomials.

[30]If G is a Lie group, its complexification is given by a complex Lie group $G^{\mathbb{C}}$ and a continuous homomorphism $h : G \to G^{\mathbb{C}}$ with the universal property that, if $f : G \to H$ is an arbitrary continuous homomorphism into a complex Lie group H, then there is a unique complex analytic homomorphism $F : G^{\mathbb{C}} \to H$ such that $f = F \circ h$.

1.3 SUSY gauge dynamics and Seiberg duality

In the following we will deal with Super Quantum ChromoDynamics (SQCD) so we begin this section with some topics that may be useful further on. First of all, SQCD is a $\mathcal{N}=1$ $SU(N)$ SYM with f flavors and the tree-level superpotential vanishes; so, classically, we have no F-flat conditions. We have f chiral superfields, Q^i_m, living in the fundamental representation of $SU(N)$ and f anti chiral superfields $\bar{Q}^m_i$ living in the anti fundamental representation of $SU(N)$; $i=1,...,f$ and $m=1,...,N$. Moreover we have a $SU(f)_L \times SU(f)_R$ non abelian global symmetry that rotates the two chiral multiplets. There are also two $U(1)$ groups: the baryonic one $U(1)_B$ and the R-symmetry one $U(1)_R$.

	$SU(N)$	$SU(f)_L$	$SU(f)_R$	$U(1)_B$	$U(1)_R$
Q	$\square$	$\square$	1	$-\frac{1}{N}$	$\frac{f-N}{f}$
$\bar{Q}$	$\bar{\square}$	1	$\bar{\square}$	$+\frac{1}{N}$	$\frac{f-N}{f}$

Table 1.1. *Table showing the matter content of the $\mathcal{N}=1$ $SU(N)$ SQCD with f flavors.*

The one loop running coupling constant g_h is given by the Renormalization Group (RG) equation

$$\beta(g_h) = \mu \frac{dg_h}{d\mu} = -\frac{b}{16\pi^2} g_h^3; \tag{1.65}$$

integrating and recalling that in our case $b = 3N - f$, we get

$$\frac{1}{g_h^2(\mu)} = -\frac{(3N-f)}{8\pi^2} ln\left(\frac{|\Lambda|}{\mu}\right) \tag{1.66}$$

where Λ is the intrinsic scale of the non abelian gauge theory that enters through dimensional trasmutation. Knowing $\frac{1}{g_h^2(\mu)}$ at one loop, we can calculate the one loop complexified gauge coupling

$$\tau_{1loop} = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g_h^2(\mu)} = ln\left(e^{\frac{i\theta_{YM}}{2\pi i}}\right) - \frac{4\pi i}{8\pi^2} ln\left(\frac{|\Lambda|}{\mu}\right)^b = \frac{1}{2\pi i} ln\left[\left(\frac{|\Lambda|}{\mu}\right)^b e^{i\theta_{YM}}\right], \tag{1.67}$$

and we can define the holomorphic intrinsic scale[31] $\tilde{\Lambda} = |\Lambda| e^{\frac{i\theta_{YM}}{b}}$ so that the holomorphic gauge coupling can be written as

$$\tau_{1loop} = \frac{b}{2\pi i} ln\left(\frac{\tilde{\Lambda}}{\mu}\right). \tag{1.68}$$

To allow non perturbative corrections we can write the most general form of τ as

$$\tau(\tilde{\Lambda}, \mu) = \frac{b}{2\pi i} ln\left(\frac{\tilde{\Lambda}}{\mu}\right) + h(\tilde{\Lambda}, \mu), \tag{1.69}$$

here $h(\tilde{\Lambda}, \mu)$ is an holomorphic function of $\tilde{\Lambda}$. First of all, $h(\tilde{\Lambda}, \mu)$ must be dimensionless; then, obviously, when $\tilde{\Lambda} \to 0$, hence in the weak coupling limit, we must recover

[31] Note that since the theta angle is periodic of period 2π we have that also the holomorphic intrinsic scale it is $\tilde{\Lambda} \to e^{\frac{2\pi i}{b}} \tilde{\Lambda}$.

the perturbative result 1.68; so $h(\tilde{\Lambda}, \mu)$ should have a Taylor series with only positive powers. Moreover, $h(\tilde{\Lambda}, \mu)$ must be invariant under the symmetry transformation $\tilde{\Lambda} \to e^{\frac{2\pi i}{b}}\tilde{\Lambda}$ and so only powers of $\tilde{\Lambda}^b$ are allowed. Due to the previous analysis, 1.69 becomes

$$\tau(\tilde{\Lambda}, \mu) = \frac{b}{2\pi i} ln\left(\frac{\tilde{\Lambda}}{\mu}\right) + \sum_{n>0} a_n \left(\frac{\tilde{\Lambda}}{\mu}\right)^{bn}; \tag{1.70}$$

hence the holomorphic gauge coupling only receives one loop corrections and non perturbative n-istanton[32] corrections, so it not runs beyond one loop. In the end, also the homolorphic gauge coupling g_h, which appears only in the combination τ, is one loop exact. It is important to underline that the holomorphic coupling is not the physical coupling. The latter is defined to be the one appearing in front of the interaction vertices when the kinetic term is canonically normalized. The physical coupling does not appear in the holomorphic combination τ, so the previous argument is no longer valid. However, it is possible to compute the perturbatively exact beta function for the physical coupling using functional integral formalism together with the superfield redefinition $V \to g_p V'$, [20],[21]; the result turns out to be anomalous and allows one to find out a relation between physical and holomorphic gauge coupling

$$\frac{1}{g_p^2} = Re\left(\frac{1}{g_h^2}\right) - \frac{2T(adj)}{8\pi^2} log(g_p) - \sum_r \frac{T(r)}{8\pi^2} ln(Z_r), \tag{1.71}$$

where with r we indicate the representations of the chiral superfields while with Z_r their renormalizations constant. Note that in the case of no flavors we obtain exactly the relation 1.58. From this relation is possible to derive the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) beta function formula

$$\beta(g_p) = -\frac{g_p^3}{16\pi^2} \frac{3T(adj) - \sum_r T(r)(1-\gamma_r)}{1 - T(adj)\frac{g_p^2}{8\pi^2}}, \tag{1.72}$$

where γ_r are the anomalous dimensions.

1.3.1 $SU(N)$ SQCD with $f < N$: Affleck-Dine-Seiberg superpotential

Let us consider SQCD with $f < N$ flavors; our goal is to write down the effective superpotential. We know that this is given by gauge invariant polynomials. From Q and $\bar{Q}$ we can construct the mason matrix $M^i_j = Q^i_m \bar{Q}^m_j$; since there is no classical relation, the classical moduli space has dimension[33] f^2. Moreover, since $f < N$, the beta function is negative and the theory is UV asymptotically free and flows to

[32] n-istanton effects are taken into account by $e^{-nS_{inst}} = \left(\frac{\tilde{\Lambda}}{\mu}\right)^{bn}$ and they are due to the theta term.

[33] In this case matter fields are $f \times N$ matrixes and so they can have at most f non zero vacuum expectation values. This implies that at a generic point of the moduli space the gauge group is spontaneously broken: $SU(N) \to SU(N-f)$. The number of unbroken generators is $(N^2-1) - ((N-f)^2 - 1) = 2Nf - f^2$. The number of D-flat directions is then the number of chiral superfields minus the number of broken generators: $2fN - (2Nf - f^2) = f^2$. This moduli space is parametrized by the $f \times f$ meson matrix.

strong coupling at low energies.
To find the effective potential, the basic requirement is that it is a singlet under the actions of the gauge group and of the $SU(f)_L \times SU(f)_R$ group. Moreover it must have R-charge two and $U(1)_B$ charge zero. Last but not least, it must be holomorphic in the chiral superfields and in the various parameters. An operator satisfying those requests is $W^\alpha W_\alpha$; we can find another gauge invariant operator, the mason matrix M, and we can construct a function of it that has the right properties: $det(M)$. Following [22], we have to use also the trick of promoting the holomorphic scale, $\tilde{\Lambda}^b$, to a spurion field in order to restore the $U(1)_A$ symmetry broken by instantons. The table with the right charges is reported below.

	$U(1)_B$	$U(1)_A$	$U(1)_R$
$det(M)$	0	$2f$	$2(f-N)$
$\tilde{\Lambda}^b$	0	$2f$	0

Table 1.2. *The table shows the charges of some operators with respect to the $U(1)$ groups.*

The superpotential can be only of the form $W_{eff} \sim \tilde{\Lambda}^{bn}(W_\alpha W^\alpha)^m det^p(M)$ with (n, m, p) to be determined in such a way that the effective superpotential has all the right traits. Turn out that the only possibilities are $(0, 1, 0)$ and $(1/(N-f), 0, -1/(N-f))$; the former is simply the tree level kinetic term while the latter is the so called Affleck-Dine-Seiberg (ADS) superpotential derived in 1984 [24]:

$$W_{ADS} = c_{N,f}\left(\frac{\tilde{\Lambda}^b}{det(M)}\right)^{\frac{1}{N-f}}. \tag{1.73}$$

The proportionality constant, $c_{N,f}$, can be exactly calculated for the case $f = N-1$ using one instanton tools[34] [23]. Moreover, there exist interesting relations between the constant $c_{N,f}$ for different number of flavors

$$c_{N,f} = (N-f)c^{\frac{1}{N-f}}, \tag{1.74}$$

hence calculating the constant for the case $f = N-1$ we can calculate it for all the values $f < N$. The calculation of the constant for $f = N-1$ gives $c_{N,N-1} = c = 1$ and so the ADS superpotential can be written as

$$W_{ADS} = (N-f)\left(\frac{\tilde{\Lambda}^b}{det(M)}\right)^{\frac{1}{N-f}}. \tag{1.75}$$

The ADS superpotential affects the moduli space; infact it depends, through the mason matrix, on scalar fields, the scalar potential V_{ADS} is minimized only at infinity[35] ($Q = \bar{Q} \to \infty$) and so all the classical moduli space is completely lifted at the quantum level and there are no stable SUSY vacua.

[34]This is suggested by the fact that for $f = N-1$ we have $W_{ADS} \sim \tilde{\Lambda}^b$ and recalling that the one instanton partition function is $e^{-S_{inst}} \sim \tilde{\Lambda}^b$ we can imagine that the ADS superpotential is generated by one instanton effect.

[35]We have not considered the fact that wave function renormalization effects can generate a Kähler potential which could give rise to local metastable minima.

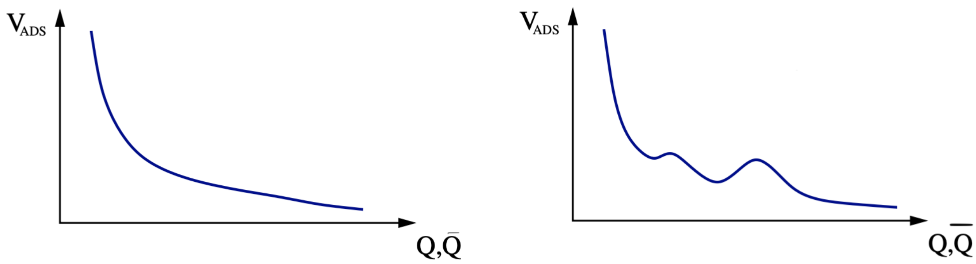

Figure 1.4. *Sketches of the behavior of the ADS scalar potential. Left: behavior of the ADS scalar potential without possible Kähler potential corrections. The moduli space is completely lifted and there are no SUSY stable vacua. Right: behavior of the ADS scalar potential with possible Kähler potential corrections that can generate metastable minima. Figures taken from [12].*

1.3.2 $SU(N)$ SQCD with $f = N$ and $f = N + 1$

For $f = N$, besides the meson matrix, there are also baryon and anti baryon operators[36]

$$B = \frac{1}{N!}\epsilon_{j_1 \dots j_N}\epsilon^{m_1 \dots m_N} Q^{j_1}_{m_1} \dots Q^{j_N}_{m_N}, \quad \bar{B} = \frac{1}{N!}\epsilon^{j_1 \dots j_N}\epsilon_{m_1 \dots m_N} \bar{Q}^{m_1}_{j_1} \dots \bar{Q}^{m_N}_{j_N}. \tag{1.76}$$

The classical moduli space is parametrized by mason matrix and baryon operators, however there is a constrain

$$det(M) - B\bar{B} = 0, \tag{1.77}$$

due to the fact that for $f = N$ then $B = det(Q)$ while $\bar{B} = \det(\bar{Q})$ and $det(M) = det(Q\bar{Q})$. This constrain reduces the dimension of the classical moduli space from $f^2 + 2$ to $f^2 + 1$[37]. This classical constrain is modified at the quantum level and the only possible strong coupling modification, compatible with the symmetries, is

$$det(M) - B\bar{B} = a\tilde{\Lambda}^{2N}. \tag{1.78}$$

with a to be determined. The constrain 1.78 can be implemented using a Lagrangia multiplayer allowing a superpotential

$$W = A\big(det(M) - B\bar{B} - a\tilde{\Lambda}^{2N}\big); \tag{1.79}$$

in order to determine the constant a we give mass to the Nth flavor [22]. What we expect is that integrating out this massive flavor, and so in the low energy limit, we recover SQCD with $f = N - 1$ flavors and hence to re-discover ADS superpotential. This is the case: integrating out Nth flavor we obtain an effective superpotential that matches the ADS one only if $a = 1$.
At the classical level, the moduli space has a conical singularity at the origin

[36] Generally speaking, for $f > N$ we can construct the baryon operators as $B_{i_1 \dots i_{f-N}} = \frac{1}{N!}\epsilon_{i_1 \dots i_{f-N} j_1 \dots j_N}\epsilon^{m_1 \dots m_N} Q^{j_1}_{m_1} \dots Q^{j_N}_{m_N}$ and $\bar{B}^{i_1 \dots i_{f-N}} = \frac{1}{N!}\epsilon^{i_1 \dots i_{f-N} j_1 \dots j_N}\epsilon_{m_1 \dots m_N} \bar{Q}^{m_1}_{j_1} \dots \bar{Q}^{m_N}_{j_N}$.

[37] This can be seen also in this way: for $f \geq N$ the gauge group is completely broken at generic point of the moduli space so the D-flat directions are $2Nf - (N^2 - 1)$. For $f = N$ we have $f^2 + 1$ D-flat directions.

$M = B = \bar{B} = 0$ while the quantum moduli space is everywhere smooth. In this case quantum corrections have deformed the moduli space. Classically the origin is part of the moduli space and so we can have zero vacuum expectation values and chiral symmetry can be unbroken; at the quantum level, origin is no longer part of the moduli space and so chiral symmetry in necessarily broken. The end of the story is that the SQCD for $f = N$ confines and the low energy dynamics is dominated by mesons and baryons, which are moduli of the space of vacua.

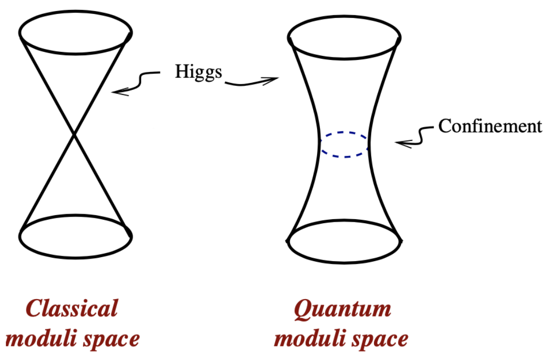

Figure 1.5. *Pictorial comparison between classical and quantum moduli space in the case of SQCD with $f = N$ flavors. Classicaly the origin is a singular part of the moduli space and in a generic point the gauge group is completely broken: we have an Higgs phase. Quantistically the origin is no longer part of the moduli space and we have an Higgs phase (for large VEVs) and a confinement phase (for VEVs near the origin). Figure taken from [12]*

We now consider the case $f = N + 1$. In this case we have the mason matrix and $N + 1$ baryons and $N + 1$ anti baryons. It is possible to show that the moduli space not only is unlifted but it also is quantum exact [25]. The interesting consequence is that the origin is in the moduli space and there we have confining dynamics without chiral symmetry breaking, a phenomenon known as s-confinement.

1.3.3 $SU(N)$ SQCD with $f \geq N + 2$: Seiberg duality

We now consider the case $f \geq N + 2$, and we need to know about the RG fixed points of the theory. First of all, looking at 1.65, we can note that for $f > 3N$ the beta function is positive and so the coupling constant flows to a weakly coupled theory at low energies: a trivial fixed point. On the other hand, for $f < 3N$ the beta function is negative and the theory is UV asymptotically free. A question arises: what happens if $f = 3N$ when the beta function vanishes? Can be this point a fixed point? The answer is yes. To see this, let us consider the NSVZ formula 1.72 for $SU(N)$ SQCD

$$\beta(g_p) = -\frac{g_p^3}{16\pi^2}\frac{3N - f(1-\gamma)}{1 - N\frac{g_p^2}{8\pi^2}}, \tag{1.80}$$

where the anomalous dimension can be computed to be

$$\gamma = -\frac{g_p^2}{8\pi^2}\frac{N^2-1}{N} + \mathcal{O}(g_p^4). \tag{1.81}$$

We expand the NSVZ formula in powers

$$\begin{aligned}\beta(g_p) &= -\frac{g_p^3}{16\pi^2}\left(3N - f\left(1+\frac{g_p^2}{8\pi^2}\frac{N^2-1}{N}+\mathcal{O}(g_p^4)\right)\right)\left(1+N\frac{g_p^2}{8\pi^2}\right) = \\ &= -\frac{g_p^3}{16\pi^2}\left[3N+3N^2\frac{g_p^2}{8\pi^2} - f - fN\frac{g_p^2}{8\pi^2} - f\frac{g_p^2}{8\pi^2}\left(\frac{N^2-1}{N}\right)+\mathcal{O}(g_p^4)\right] = \\ &= -\frac{g_p^3}{16\pi^2}\left[3N - f + \frac{g_p^2}{8\pi^2}\left(3N^2 - 2fN + \frac{f}{N}\right)\right] + \mathcal{O}(g_p^7)\end{aligned} \tag{1.82}$$

and we consider a f arbitrarily close to $3N$ defining

$$\epsilon := 3 - \frac{f}{N} << 1. \tag{1.83}$$

With the previous definition we can rewrite the expansion of the NSVZ formula as

$$\beta(g_p) = -\frac{g_p^3}{16\pi^2}\left[\epsilon N - \frac{g_p^2}{8\pi^2}[3(N^2-1)+\mathcal{O}(\epsilon)]\right] + \mathcal{O}(g_p^7); \tag{1.84}$$

hence we can get a vanishing beta function if the part inside square brackets vanishes up to ϵ corrections

$$\epsilon N - \frac{g_p^2}{8\pi^2}3(N^2-1) = 0 \Rightarrow g_p^* = \frac{8\pi^2}{3}\frac{N}{N^2-1}\epsilon. \tag{1.85}$$

This is the SQCD analogue of the Banks-Zaks (BZ) fixed point in QCD [26].

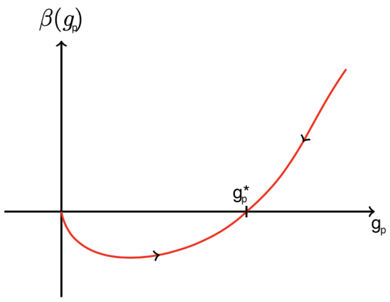

Figure 1.6. *Schematic behavior of the beta function. g_p^* is the Banks-Zaks non trivial fixed point.*

Since the theory is supersymmetric, if it flows to a non-trivial fixed point, the quantum theory is both supersymmetric and conformal, and the supersymmetry algebra is enlarged to its conformal extension, known as superconformal algebra (see appendix B for a short review of conformal and superconformal algebra). There is more: Seiberg proposes that the Banks-Zaks fixed point really exists in the conformal window defined by the range $\frac{3}{2}N < f < 3N$ [27] and we call this Seiberg fixed point. A second proposal due to Seiberg is that the IR physics of SQCD, called in this context electric theory (eSQCD), for $f \geq N+2$ has an equivalent description in terms of another SUSY gauge theory called magnetic theory (mSQCD). The two theories are dual. They are not equivalent in the UV regime and also along the RG flow but they are equivalent in the IR: they flow at the same IR non trivial fixed point. In order to better understand this conjecture we consider the mason, baryon and antibaryion operators for the eSQCD with $f \geq N+2$. The baryon and the antibaryon operators have $f - N := \tilde{N}$ free indexes and we can think of them, respectively, as bound state of $\tilde{N}$ some new fields q and $\bar{q}$ transforming in the fundamental and the antifundamental of some SYM theory with gauge group $SU(\tilde{N}) = SU(f-N)$; moreover the mSQCD must have a non vanishing potential of the form $W = \lambda \tilde{M}^i_j q^i \bar{q}_j$ where $\tilde{M}^i_j$ is the dual mason matrix. So the baryon and the antibaryon operators and the mason matrix of the eSQCD have a dual description in term of the fields present in the mSQCD

$$\begin{gathered} B_{i_1 \dots i_{f-N}} \sim \epsilon^{m_1 \dots m_N} Q^{j_1}_{m_1} \dots Q^{j_N}_{m_N} \overset{D}{\longleftrightarrow} \tilde{B}_{i_1 \dots i_{f-N}} \sim \epsilon^{m_1 \dots m_{\tilde{N}}} q^{j_1}_{m_1} \dots q^{j_{\tilde{N}}}_{m_{\tilde{N}}}; \\ \bar{B}^{i_1 \dots i_{f-N}} \sim \epsilon_{m_1 \dots m_N} \bar{Q}^{m_1}_{j_1} \dots \bar{Q}^{m_N}_{j_N} \overset{D}{\longleftrightarrow} \tilde{\bar{B}}^{i_1 \dots i_{f-N}} \sim \epsilon_{m_1 \dots m_{\tilde{N}}} \bar{q}^{m_1}_{j_1} \dots \bar{q}^{m_{\tilde{N}}}_{j_{\tilde{N}}} \\ M^i_j \overset{D}{\longleftrightarrow} \tilde{M}^i_j; \end{gathered} \tag{1.86}$$

where D stands for "DUAL". Beside this, for mSQCD we can recycle the beta function 1.65 replacing N with $f - N$, so we get

$$\beta(g_h) = \mu \frac{dg_h}{d\mu} = -\frac{\tilde{b}}{16\pi^2} g_h^3 \tag{1.87}$$

where we have defined

$$\tilde{b} = 3\tilde{N} - f = 3(f-N) - f = 2f - 3N. \tag{1.88}$$

Hence the mSCQD is UV asymptotic free when $f > \frac{3}{2}N$ and IR asymptotic free when $f < \frac{3}{2}N$. Moreover if we look at the conformal window of mSQCD we obtain

$$\frac{3}{2}\tilde{N} < f < 3\tilde{N} \Rightarrow \frac{3}{2}(f-N) < f < 3(f-N) \Rightarrow \frac{3}{2}N < f < 3N; \tag{1.89}$$

this is exactly the conformal window for eSQCD: the conformal windows of the two dual theories are the same and so the two theories in this window flow to the same IR non trivial fixed point. It is also interesting to note that when the eSQCD is UV free the mSQCD is IR free and viceversa. This is to be considered the real power of Seiberg duality: just when one description of the theory is becoming non perturbative the other one is returning to perturbativity. A brief recap on Seiberg duality is reported in the following table.

	$f < \frac{3}{2}N$	$\frac{3}{2}N < f < 3N$	$f > 3N$
eSQCD	UV free	DUAL: same IR fixed point	IR free
mSQCD	IR free	DUAL: same IR fixed point	UV free

Table 1.3. *Schematic recap of Seiberg duality for $SU(N)$. In this case eSQCD is a $SU(N)$ SYM theory with f flavors while mSQCD is a $SU(\tilde{N}) = SU(f-N)$ SYM theory with f flavors. In the conformal window the two theories are dual and they flow to the same IR fixed point. In the regions where one theory is strong coupled the other is weakly coupled.*

1.3.4 SUSY gauge dynamics and Seiberg duality for other groups: $SO(N)$ and $USp(N)$

What we have done in these paragraphs can be generalized to SUSY gauge theories with other gauge groups. We now see this generalization to the case of $SO(N)$ and $USp(N)$ gauge group; this will reveal useful later on.
We consider $SO(N)$ and $USp(N)$ SYM gauge theories with f flavors and with matter in the fundamental representation $Q_1^{a_1}, ..., Q_f^{a_f}$, where a_i are the gauge group indexes. Obviously for the case of $USp(N)$ we must have $N = 2k$ and, less obvious, also f even[38]. For these theories the one loop beta function coefficient is known

$$b_{SO(N)} = 3(N-2) - f, \; b_{USp(N)} = 3(N+2) - f. \tag{1.90}$$

$SO(N)$ SQCD

We can define the mason matrix $M_{ij} = \delta_{ab} Q_i^a Q_j^b$ and it is symmetric in the flavor indexes i, j.
For $f < N - 2$ there is a dynamically generated ADS-like superpotential

$$W_{ADS} = (N - f - 2)\Big(\frac{\tilde{\Lambda}^{b_{SO(N)}}}{det(M)}\Big)^{\frac{1}{N-f-2}}, \tag{1.91}$$

this superpotential lifts the classical moduli space of the theory at the quantum level.
For $f \geq N$ Seiberg duality is possible; the dual theory has gauge group $SO(\tilde{N}) = SO(f - N + 4)$, contains f chiral superfields $q_1, ..., q_f$ and the dual meson matrix $\tilde{M}_{ij}$. Moreover the dual theory has a superpotential $W = \tilde{M}_{ij} q_i q_j$. As for $SU(N)$ SQCD Seiberg duality we have a duality map between the gauge invariant operators of the two dual theory

$$\begin{aligned}
B_{[i_1,...,i_N]} = Q_{[i_1} ... Q_{i_N]} &\overset{D}{\longleftrightarrow} \epsilon_{i_1,...,i_f} \tilde{h}^{[i_1,...,i_{\tilde{N}-4}]} = \epsilon_{i_1,...,i_f} \tilde{W}_\alpha^2 q^{[i_1} ... q^{i_{\tilde{N}-4}]};\\
h_{[i_1,...,i_{N-4}]} = W_\alpha^2 Q_{[i_1} ... Q_{i_{N-4}]} &\overset{D}{\longleftrightarrow} \epsilon_{i_1,...,i_f} \tilde{B}^{[i_1,...,i_{\tilde{N}}]} = \epsilon_{i_1,...,i_f} q^{[i_1} ... q^{i_{\tilde{N}}]};\\
H_{\alpha,[i_1,...,i_{N-2}]} = W_\alpha Q_{[i_1} ... Q_{i_{N-4}]} &\overset{D}{\longleftrightarrow} \epsilon_{i_1,...,i_f} \tilde{H}^{\alpha,[i_1,...,i_{\tilde{N}-2}]} = \epsilon_{i_1,...,i_f} \tilde{W}^\alpha Q^{[i_1} ... Q^{i_{\tilde{N}-4}]};\\
M_{ij} &\overset{D}{\longleftrightarrow} \tilde{M}_{ij};
\end{aligned} \tag{1.92}$$

where D stands for "DUAL" while the square brackets indicates antisymmetrization on the indexes. B is the baryon operator in the original theory and $\tilde{B}$ is the baryon operator in the dual theory. Furthermore, in $SO(N)$ SQCD are present the so called hybrids h and H_α; $\tilde{h}$ and $\tilde{H}^\alpha$ are the hybrids of the dual theory. W_α and $\tilde{W}^\alpha$, respectively for the two theories, are defined by 1.54. The conformal window, in which both the dual theories flow to the same IR non trivial fixed point is

$$\frac{3}{2}(N-2) < f < 3(N-2). \tag{1.93}$$

To treat $f = N - 2$ and $f = N - 1$ cases goes beyond the scope of this paragraph.

[38]This is due to the Witten anomaly [28]: non perturbative anomaly related to the global topological structure of the gauge groups, affecting all the gauge theories whose gauge group has a non trivial fourth homotopy group. Unitary groups are free from this anomaly, as well as orthogonal groups; for symplectic groups this anomaly is avoided if the number of flavors is even.

	$f < \frac{3}{2}(N-2)$	$\frac{3}{2}(N-2) < f < 3(N-2)$	$f > 3(N-2)$
eSQCD	UV free	DUAL: same IR fixed point	IR free
mSQCD	IR free	DUAL: same IR fixed point	UV free

Table 1.4. *Schematic recap of Seiberg duality for $SO(N)$. In this case eSQCD is a $SO(N)$ SYM theory with f flavors while mSQCD is a $SO(\tilde{N}) = SO(f-N+4)$ SYM theory with f flavors. In the conformal window the two theories are dual and they flow to the same IR fixed point. In the regions where one theory is strong coupled the other is weakly coupled.*

$USp(N)$ SQCD

We can define the mason matrix $M_{ij} = Q_i^a \Omega_{ab} Q_j^b$ which is antisymmetric in the flavor indexes i, j; Ω is the standard symplectic matrix. In this theory there are no independent baryon operators since the Levi-Civita tensor can be rewritten in terms of the standard symplectic matrix Ω and so baryon operators are functions of the mesons.

For $f \leq N$ there is a dynamically generated ADS-like superpotential

$$W_{ADS} = (N-f+2)\left(\frac{\tilde{\Lambda}^{b_{USp(N)}}}{Pf(M)}\right)^{\frac{1}{N-f+2}}, \tag{1.94}$$

where $Pf(\cdot)$ is the pfaffian[39]; also in this case, the classical moduli space is lifted at the quantum level.

For $f \geq N+4$ Seiberg duality holds. The dual theory has gauge group $USp(\tilde{N}) = USp(N-f-4)$ and contains f chiral superfields $q_1, ..., q_f$. It exists also a superpotential, $W = \tilde{M}_{ij} q_i q_j$ where $\tilde{M}_{ij}$ is the dual mason matrix. In this case the duality map is quite trivial

$$M_{ij} \overset{D}{\longleftrightarrow} \tilde{M}_{ij} \tag{1.95}$$

where D stands for "DUAL". The conformal window, in which both side of the duality flow to the same IR non trivial fixed point, is

$$\frac{3}{2}(N+2) < f < 3(N+2). \tag{1.96}$$

For $f = N+2$ one finds confinement while for $f = N+4$ one finds s-confinement.

	$f < \frac{3}{2}(N+2)$	$\frac{3}{2}(N+2) < f < 3(N+2)$	$f > 3(N+2)$
eSQCD	UV free	DUAL: same IR fixed point	IR free
mSQCD	IR free	DUAL: same IR fixed point	UV free

Table 1.5. *Schematic recap of Seiberg duality for $USp(N)$. In this case eSQCD is a $USp(N)$ SYM theory with f flavors while mSQCD is a $USp(\tilde{N}) = USp(f-N-4)$ SYM theory with f flavors. In the conformal window the two theories are dual and they flow to the same IR fixed point. In the regions where one theory is strong coupled the other is weakly coupled.*

[39] The pfaffian is not zero only for an antisymmetric matrix and satisfies the relation $Pf^2(A) = det(A)$. In our case the mason matrix is antysimmetric in flavor indexes and f can be only even.

Chapter 2

String theory

Nowadays we understand that there are four fundamental forces or interactions: strong and weak interaction, electromagnetism and gravity. Moreover physicists believe that we understand three out four fundamental interactions, in the sense that we know what the lagrangian is and, in principle, we know how to calculate, using this lagrangian, well defined predictions. As for gravity, we only understand it partially. We know it classically, thanks to General Relativity (GR) and we also know it quantistically, in the sense that we are able to include quantum effect into gravity as long as we do not ask ourselves questions about what is going on at distances less than the Planck length $l_P \sim 1,6 \times 10^{-35} m$.
The inclusion of quantum effect in gravity leads to interesting discoveries such as the Hawking radiation [29],[30],[32], that is the fact that black holes are not really black but they emit thermal radiation with temperature T_H that in the case of a Schwarzschild black hole is

$$T_H = \frac{\hbar c^3}{8\pi G k_B M} \simeq 6 \times 10^{-8} \left(\frac{M_\odot}{M}\right) K$$

where M is the black hole mass. The fact that black holes emit radiation leads to the information loss paradox [31],[33]: if a particle falls into the black hole and black hole emits thermal radiation then there must be a transition between a pure state, the particle that has fallen, and a statistical mixture, the emitted thermal radiation with consequent loss of information. This would be a violation of unitarity of quantum mechanics if the black hole evaporated completely; today we do not have an answer to this paradox but only some hypotheses. Another important consequence of radiation emitted by black holes is the possibility to formulate a black holes thermodynamics [32],[33]: a list of four principles that strongly recall ordinary thermodynamics. For example the first of this principles says that for a Kerr-Newman[1] black hole the variation of the mass/energy is, in natural units,

$$dM = \frac{\mathcal{K}}{8\pi} dA + \Omega dJ + \Phi dQ$$

where M is the mass of the black hole, $\mathcal{K}$ is its surface gravity, Ω is its angular velocity and Φ is its electrostatic potential.

[1] Kerr-Newman black hole is the most general black hole solution for $4D$ general relativity. It is an axial symmetric rotating charged black hole.

Last but not least, the radiation emitted by black holes leads to their evaporation according to the law

$$M(t) = M\left(1 - \frac{t}{t_{ev}}\right)^{\frac{1}{3}};\ t_{ev} = \frac{5120\pi M^3 G^2}{\Gamma \hbar c^4} \simeq \frac{2,1}{\Gamma} \times 10^{67}\left(\frac{M}{M_{\odot}}\right)^3 years;$$

where Γ is a greybody correction factor. This number is incredibly big when compared with the age of the universe $t_u \simeq 13,7 \times 10^9 years$. However, is probably that this evaporation cannot last until the black hole is completely evaporated: when the black hole dimensions become comparable with the Planck scale, full quantum gravity effects enter in the game.
In these few lines we have seen that the inclusion of the quantum effect[2] in gravity implies a very interesting physics. Beside this, we are not able to construct a full theory of quantum gravity; infact when we go down under the Planck scale and we consider general relativity as a QFT, we find a non renormalizable theory. This means that below the Planck scale we need more and more parameters to absorb the infinity that occur in the theory. In the end we need an infinite number of parameter and hence an infinite number of measurement; GR is useless at this scale, probably it is not the fundamental theory of gravity. Finding a Quantum Gravity (QG) theory would allow us to investigate and try to answer questions concerning black hole gravitational singularities and cosmological ones such as, what happens or what is the Big Bang?
Several proposals have been made for a QG theory and the two main ways are string theory and Loop Quantum Gravity. The rest of this chapter will focus on string theory so here we will briefly discuss Loop Quantum Gravity. This theory is born in 1986 with the work of Abhay Ashtekar [34],[35] and in 1994 Carlo Rovelli and Lee Smolin showed that the quantum operators of the theory associated to area and volume have a discrete spectrum, so geometry is quantized [36].
Back to string theory, the main idea is that we replace point particles, zero dimension objects, with string hence with a one dimension objects whose characteristic length is $l_s \simeq 10^{-35}m$ and so order of the Planck scale. This new fundamental objects, the strings, can oscillate and their oscillations are interpreted like particles: different states of oscillation are different particles. In this sense, in string theory gravity emerges quite naturally as a traceless symmetric 2-form $g_{\mu\nu}$: the graviton.
The first formulation of string theory was the bosonic string theory developed in the sixties. It was born in the context of adronic physics [37] in which a lot of strong interacting particles were found. The name "bosonic string theory" is due to the fact that its spectrum contains only boson and so this theory is not able to describe leptons or quarks and hence it could not be the right theory; moreover turn out that the vacuum state is a tachyon[3] and so is unstable. One first important consequence of the fact the the strings are one dimensional objects is that they describe, in the space-time, a surface, the so-called worldsheet, and not a line like point particles; this implies, for example, that Feynman diagrams are replaced by diagrams with

[2] In Hawking temperature and time evaporation formulas we have inserted all the physical constant in order to underline their dependence by $\hbar$: these are quantum effects.

[3] A tachyon is a state with imaginary mass. Looking at the relation $E = m\gamma$ in which E is a real quantity we see that if m is imaginary then also the Lorentz factor, γ, should be. This implies $v > 1$ and so a tachyon must move faster than light.

surface and turn out that a fixed order in perturbative expansion contains surfaces with well defined and fixed genus. When SUSY was discovered, a new kind of theory

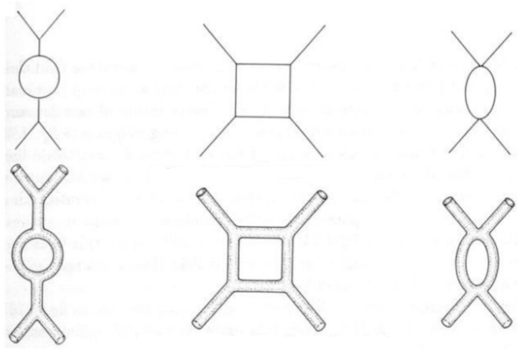

Figure 2.1. *Difference between diagrammatic development in the case of point particles (above) and strings (below). Figure taken from [40] vol 1.*

was developed: superstring theory. This theory is able to describe both bosons and fermions and, thanks to the so-called Gliozzi-Scherk-Olive (GSO) projection, is tachyon free.
In the mid eighties there was some confusion; it was found that there are five different consistent string theories:

- Type I: it is the only one in which strings are unoriented and which contains not only closed strings, but also open strings.
- Type IIA: it contains closed string and is a not chiral theory;
- Type IIB: it contains closed string and is a chiral theory
- $SO(32)$ heterotic: this is a theory of closed string and it is a heterosis theory, in which counter-clockwise vibrational patterns live in 26 dimensions and clockwise patterns live in 10 dimensions. Such combination is possible because the right and left movers are independent from each other. The extra 16 left mover dimensions provide the gauge group for the resulting 10 dimensional theory. In this case the consistent gauge group is $SO(32)$.
- $E_8 \times E_8$ heterotic: similar to the $SO(32)$ heterotic string theory but the consistent gauge group for the 10 dimensional theory is $E_8 \times E_8$[4].

Many physicists showed that the theories were related in intricate and nontrivial ways by a web of duality, and in 1995, Edward Witten conjectured the existence of a theory, called M-theory, that try to unify all consistent versions of superstring theory [63].

It is important to underline an interesting characteristic of string theory. SM contains a lot of free parameters such as quark and lepton masses while string theory contains only one parameter: the string length l_s. However, this is non really true:

[4] E_8 is a Lie group with rank 8 and dimension 248. It is part of the so-called exceptional groups of which it is the most complicated representative.

infact to describe a theory we need to know the ground state and turn out that there is a landscape of vacua for string theory.
In the following we will study the basic of string theory [38],[39],[40],[44]. studying boson string theory and type IIA and IIB superstring theory. We will begin with bosonic string theory that is useful in order to understand a lot of features of string theory in general, then we will move to superstring theories adding SUSY at the bosonic string. At the end of this chapter we will study the web of duality that interconnects different superstring theories and we will encounter D-branes: objects that will be essential for the Anti de Sitter/Conformal Field Theory (AdS/CFT) correspondence formulation.

2.1 Bosonic string theory

We know that, in natural units, the point particle action is $S_0 = -m \int ds$ where m is the mass particle while ds is the worldline element. The most obvious generalization of this zero dimensional object action is to consider a p-brane action in a D-dimensional background space with metric $g_{\mu\nu}$

$$S_p = -T_p \int d\mu_p = T_p \int \sqrt{-det(H_{\alpha\beta}(X))} d^{p+1}\sigma, \tag{2.1}$$

where $d\mu_p := \sqrt{-det(H_{\alpha\beta}(X))} d^{p+1}\sigma$ is the $(p+1)$-dimensional volume element and T_p is the p-brane tension, its units are $[T_p] = \frac{kg}{vol}$. Let us explain better 2.1. We parameterize the $p+1$ worldsurface generated by the p-brane with $p+1$ parameters $\sigma^0 = \tau, \sigma^1, ..., \sigma^p$ with τ time-like coordinate while all the others are space-like. The embedding of the p-brane into the D-dimensional background space-time is given by a set of fields $X^\mu(\tau, \sigma^1, ..., \sigma^p) := X^\mu(\tau, \vec{\sigma})$ with $\mu = 0, ..., D-1$; from this embedding of the p-brane into the D-dimensional background space, the induced metric $H_{\alpha\beta}(X) := \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} g_{\mu\nu}(X)$ with $\alpha, \beta = 0, ..., p-1$ arises.
We focus on strings, so on 1-brane: the action 2.1 becomes

$$S = -T \int \sqrt{-det(H_{\alpha\beta}(X))} d\tau d\sigma, \tag{2.2}$$

now, if we assume that the background space is flat and we use 2.2 and the definition of the induced metric, we get the Nambu-Goto (NG) string action[5]

$$S_{NG} = -T \int \sqrt{(\dot{X} \cdot X')^2 - (\dot{X}^2)(X')^2} d\tau d\sigma, \tag{2.3}$$

where the prime indicates derivative with respect to σ while the dot derivative with respect to τ; this is a convention that we use in all chapter. Unfortunately, in the Nambu-Goto action we have to deal with a square root. In order to avoid this we define an auxiliary field: the worldsheet metric $h_{\alpha\beta}(\tau, \sigma)$. This field is another metric of the worldsheet which is different from the induced metric $H_{\alpha\beta}(X)$ and using it we

[5]It can be shown [41] that Nambu-Goto action can be interpreted as the area of the worldsheet and, since equation of motion comes out minimizing the action, one can think the equation of motion as the worldsheet of smallest area.

can define a new string action classically completely equivalent[6] to the Nambu-Goto one

$$S_P = -\frac{T}{2}\int \sqrt{-h}h^{\alpha\beta}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}g_{\mu\nu} \tag{2.4}$$

where h is the determinant of the worldsheet metric. Action 2.4 is called Polyakov action. We now choose our background space-time to be minkowskian and so $g_{\mu\nu} = \eta_{\mu\nu}$

$$S_P = -\frac{T}{2}\int \sqrt{-h}h^{\alpha\beta}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu} \tag{2.5}$$

This is Polyakov action in Minkowski background space-time and it is invariant under the joint transformations $\delta X^\mu = \omega^\mu_\nu X^\nu + b^\mu$ and $\delta h^{\alpha\beta} = 0$; these are Poincaré transformations and so Polyakov action is Poincaré invariant, as required for all good actions. Moreover, the action 2.5 has two local symmetries: diffeomorphisms and Weyl invariance. The former is the invariance under reparametrization obtained from a diffeomorphism $f(\sigma)$ while the latter is the invariance under transformation of the form $h'_{\alpha\beta}(\tau,\sigma) = e^{2\phi(\sigma)}h_{\alpha\beta}(\tau,\sigma)$ called Weyl transformations. In both this kind of transformations the embedding functions are untouched. An important consequence of the Weyl invariance is that the stress energy tensor, $T_{\alpha\beta} := -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S_P}{\delta h_{\alpha\beta}}$, is traceless: considering an infinitesimal Weyl transformation we have

$$\begin{aligned} 0 = \delta S_P := \int \frac{\delta S_P}{\delta h^{\alpha\beta}}\delta h^{\alpha\beta} = -\frac{T}{2}\int d\tau d\sigma\sqrt{-h}T_{\alpha\beta}\delta h^{\alpha\beta} = \\ = -\frac{T}{2}\int d\tau d\sigma\sqrt{-h}T_{\alpha\beta}(-2\phi)h^{\alpha\beta}, \end{aligned} \tag{2.6}$$

and hence $T_{\alpha\beta}h^{\alpha\beta} = T^\alpha_\alpha = 0$.
The real power of the local invariances of the Polyakov action is that we can always fix a gauge in which the worldsheet metric is flat. Indeed since $h_{\alpha\beta}$ is a 2×2 symmetric matrix has only three independent elements, hence we can constrain them to be $h_{10} = h_{01} = 0$ and $h_{00} = -h_{11}$ and then using a Weyl transformation we can recast the metric to be flat $h_{\alpha\beta} = \eta_{\alpha\beta}$. However there is a caveat: since gauge invariance are local, the previous analysis is valid only locally. In general we are not able to extend the flat gauge on the whole worldsheet; this is possible only if the worldsheet has trivial Euler characteristic[7]. In terms of the flat gauge the Polyakov action appears as

$$S_P = \frac{T}{2}\int ((\dot{X})^2 - (X')^2)d\tau d\sigma. \tag{2.7}$$

2.1.1 Equations of motion and boundary conditions

Consider the Polyakov action 2.7, setting its variation with respect to X^μ to zero and setting to zero the variation of X^μ at the boundary of τ we get the equations of

[6] This can be seen computing the equation of motion for the auxiliary field $h_{\alpha\beta}(\tau,\sigma)$. In the end what one finds for the equation of motion of $h_{\alpha\beta}(\tau,\sigma)$ is that $\frac{1}{2}\sqrt{-h}h^{\alpha\beta}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta} = \sqrt{-det(H_{\alpha\beta})}$. Substituting them back one finds the equivalence.

[7] This is because to extend the flat metric globally it must exist a flat coordinate system that cover the whole worldsheet implying that the Ricci scalar is zero. Now due to the Gauss-Bonnet theorem in two dimension, $\chi = \frac{1}{4\pi}\int\sqrt{-h}Rd\tau d\sigma$, we must have vanishing Euler characteristic.

motion of the string

$$(-\partial_\tau^2 + \partial_\sigma^2)X^\mu = 0 \tag{2.8}$$

along with a boundary term that we can make null with the choice of the boundary conditions for σ that determine what kind of string we have: closed or open strings. Conventionally, we take the endpoints of the string, respectively, in $\sigma = 0$ and $\sigma = \pi$ hence $\sigma \in [0, \pi]$.
If we impose periodic boundary conditions

$$X^\mu(\tau, \sigma = \pi) = X^\mu(\tau, \sigma = 0) \tag{2.9}$$

we talk about closed strings while if we want to talk about open strings we have two possible boundary conditions:

- Dirichlet boundary condition (DBC): we set the values of the two endpoints of the string to two constants, $X^\mu(\tau, \sigma = 0) = X_0^\mu$ and $X^\mu(\tau, \sigma = \pi) = X_\pi^\mu$. Dirichlet boundary condition fixes the endpoints and so breaks Poincarè invariance in the directions in which this boundary condition is applied.

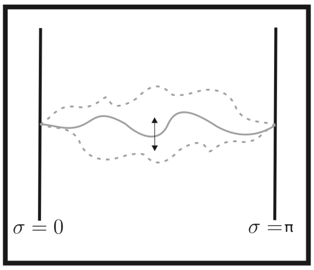

Figure 2.2. *Dirichlet boundary condition: the string can oscillate but its endpoints are fixed. Figure taken from [38].*

- Neumann boundary condition (NBC): we set the derivative of X^μ with respect to σ zero at the two endpoints of the string, $\partial_\sigma X^\mu\big|_{\sigma=\pi} = \partial_\sigma X^\mu\big|_{\sigma=\pi} = 0$. Note that this boundary condition preserve Poincaré invariance

$$\partial_\sigma X'^\mu\big|_{\sigma=0,\pi} = \partial_\sigma(\omega^\mu_\nu X^\nu + b^\mu)\big|_{\sigma=0,\pi} = \omega^\mu_\nu \partial_\sigma X^\mu\big|_{\sigma=0,\pi} = 0.$$

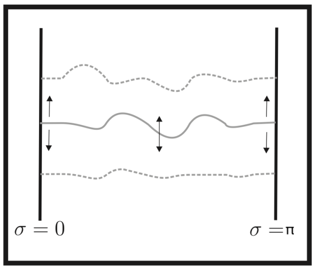

Figure 2.3. *Neumann boundary condition: the string can oscillate and its endpoints can move as long as their derivatives vanish. Figure taken from [38].*

The possibility to impose for some X^μ, for example $p+1$ of them[8], the NBCs and hence for the rest $D-p-1$ the DBCs, leads to the existence of a $p+1$ dimensional hyperplane called Dp-brane, or simply D-brane, on which the strings end and can move. The presence of this Dp-brane is what breaks Poincaré invariance. Below we have a picture of strings that end on Dp-branes.

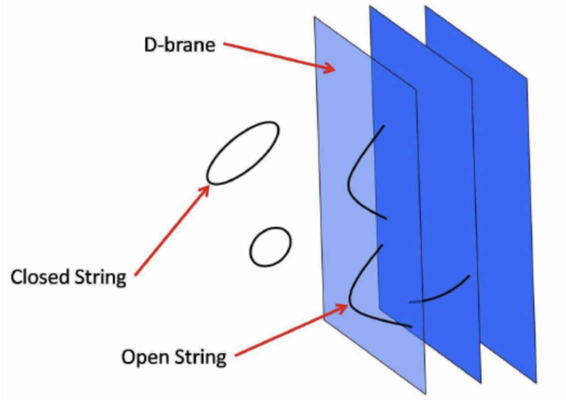

Figure 2.4. *Example of strings that ends on a Dp-brane; we note also the case in which the string ends on two different Dp-branes. Figure taken from [38].*

We know that in addition to the string equation we must impose the equations of motion of the worldsheet metric; these are, in the generic gauge

$$0 = T_{\alpha\beta} = (\partial_\alpha X^\mu)(\partial_\beta X^\nu)g_{\mu\nu} - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}(\partial_\gamma X^\mu)(\partial_\delta X^\nu)g_{\mu\nu}, \tag{2.10}$$

and they become

$$0 = T_{00} = T_{11} = \frac{1}{2}((\dot{X})^2 + (X')^2),\ 0 = T_{01} = T_{10} = \dot{X}\cdot X' \tag{2.11}$$

in the flat gauge. They are called Virasoro constrains
At this point we want to solve the equations of motion of the string; obviously we have to distinguish between closed and open strings and between DBC and NBC. First of all we recast all in the light-cone coordinates, $\sigma^\pm$, defined by

$$\sigma^\pm := \tau \pm \sigma. \tag{2.12}$$

In these coordinates the string equations of motion are written as

$$\partial_+\partial_- X^\mu = 0 \tag{2.13}$$

and the Virasoro constrains become

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu = 0,\ T_{--} = \partial_- X^\mu \partial_- X_\mu = 0; \tag{2.14}$$

[8] p indicates the number of spatial dimensions; the +1 is because the time coordinate of the background space $X^0(\tau,\sigma)$ can not have DBC since time flows.

where with + we indicate the coordinate σ^+ and with − we indicate the coordinate σ^-. The equations 2.13 is a set of D $1+1$ d'Alembert equations in the variables $\sigma^\pm$, hence we can find a solution of the form

$$X^\mu(\sigma^+,\sigma^-) = \underbrace{X^\mu_R(\sigma^-)}_{right\ mover} + \underbrace{X^\mu_L(\sigma^+)}_{left\ mover}, \tag{2.15}$$

where the right and left movers can be found imposing the appropriate boundary conditions:

- Closed strings: imposing DBCs we found the particular expansion corresponding to closed strings

$$\begin{aligned} X^\mu_R &= \frac{1}{2}x^\mu + \frac{1}{2}l_s^2\sigma^- p^\mu + \frac{i}{2}l_s\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-2in\sigma^-}, \\ X^\mu_L &= \frac{1}{2}x^\mu + \frac{1}{2}l_s^2\sigma^+ p^\mu + \frac{i}{2}l_s\sum_{n\neq 0}\frac{\beta^\mu_n}{n}e^{-2in\sigma^+}, \end{aligned} \tag{2.16}$$

  where x^μ and p^μ are the coordinates center of mass and the momentum of the center of mass of the string. $l_s = \sqrt{2\alpha'}$ is the typical string length and $\alpha' = \frac{1}{2\pi T}$ is the so-called Regge slope. Knowing the right and left movers we know the total solution

$$X^\mu(\tau,\sigma) = x^\mu + l_s^2\tau p^\mu + \frac{i}{2}l_s\sum_{n\neq 0}\frac{1}{n}(\alpha^\mu_n e^{2in\sigma} + \beta^\mu_n e^{-2in\sigma})e^{-2in\tau}. \tag{2.17}$$

  The first two terms describe the motion of the center of mass of the string while the term in summation describes the oscillations of the string. So a closed string oscillates and in the meantime it moves at the speed of light.

- Open string: for an open string we can impose DBC or NBC and the the two expansions are, obviously, different

$$\begin{aligned} NBC &\Rightarrow X^\mu(\tau,\sigma) = x^\mu + l_s^2\tau p^\mu + il_s\sum_{m\neq 0}\frac{\alpha^\mu_m}{m}e^{-im\tau}cos(m\sigma), \\ DBC &\Rightarrow X^\mu(\tau,\sigma) = x^\mu_0 + \frac{\sigma}{\pi}(x^\mu_\pi - x^\mu_0) + \sum_{m\neq 0}\frac{\alpha^\mu_m}{m}e^{-im\tau}sin(m\sigma); \end{aligned} \tag{2.18}$$

  also this time we have the term of motion of the center of mass and the oscillating term.

In the following we will consider open strings with only NBCs, and we postpone the discussion on the presence of Dp-branes to the paragraph 2.3.

At this point, is useful to define the following objects

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{m-n}\cdot\alpha_n,\ \tilde{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\beta_{m-n}\cdot\beta_n; \tag{2.19}$$

and they satisfy a well defined algebraic structure

$$\{L_m, L_n\} = i(m-n)L_{m+n},\ \{\tilde{L}_m,\tilde{L}_n\} = i(m-n)\tilde{L}_{m+n}, \tag{2.20}$$

called Witt algebra. Obviously, for a closed strings we will need both L_m and $\tilde{L}_m$ while for an open string only the first. These objects are important because if we rewrite the Virasoro constrains in term of the oscillation modes we will find that it must be $L_m = \tilde{L}_m = 0\ \forall m$ to make sure that Virasoro constrains are satisfied. Moreover, thanks to these objects we can derive a mass formula for both open and closed strings. In the case of an open string this is the way[9]

$$\begin{aligned} 0 = L_0 = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{-n}\cdot\alpha_n = \underbrace{\frac{1}{2}\alpha_0^2}_{=-M^2\alpha'} + \frac{1}{2}\Big(\sum_{n=-\infty}^{-1}\alpha_{-n}\cdot\alpha_n + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n\Big) = \\ = -M^2\alpha' + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n \Rightarrow M^2 = \frac{\sum_{n>1}\alpha_{-n}\cdot\alpha_n}{\alpha'}; \end{aligned} \tag{2.21}$$

in the case of closed strings we get

$$M^2 = \frac{2\sum_{n>1}(\alpha_{-n}\cdot\alpha_n + \beta_{-n}\cdot\beta_n)}{\alpha'}. \tag{2.22}$$

These are the mass-shell conditions for open and closed strings and tell us the mass corresponding to a certain classical string state.
These two mass relations conclude our study of classical bosonic string; now we move to quantize this theory.

2.1.2 Quantization of bosonic string

There are many ways to quantize a theory. One of them is canonical quantization in which we have to impose some commutation relations. In the following we indicate operators with an hat.
The commutation relations satisfied by the oscillation modes are[10]

$$[\hat{\alpha}_m^\mu, (\hat{\alpha}_{-n}^\nu)^\dagger] = m\eta^{\mu\nu}\delta_{m,-n},\ [\hat{\beta}_m^\mu, (\hat{\beta}_{-n}^\nu)^\dagger] = m\eta^{\mu\nu}\delta_{m,-n},\ [\hat{\alpha}_m^\mu, \hat{\beta}_n^\nu] = 0, \tag{2.23}$$

hence we can define two sets of creation and annihilation operators

$$\begin{aligned} \hat{a}_m^\mu &:= \frac{1}{\sqrt{m}}\hat{\alpha}_m^\mu,\ (\hat{a}_m^\mu)^\dagger = \frac{1}{\sqrt{m}}(\alpha_{-m}^\mu)^\dagger; \\ \hat{b}_m^\mu &:= \frac{1}{\sqrt{m}}\hat{\beta}_m^\mu,\ (\hat{b}_m^\mu)^\dagger = \frac{1}{\sqrt{m}}(\beta_{-m}^\mu)^\dagger; \end{aligned} \tag{2.24}$$

then the only non vanishing commutation relations that they satisfy are

$$[\hat{a}_m^\mu, (\hat{a}_n^\nu)^\dagger] = [\hat{b}_m^\mu, (\hat{b}_n^\nu)^\dagger] = \eta^{\mu\nu}\delta_{m,n}. \tag{2.25}$$

The relations 2.25 are very similar to the commutation relations in Gupta-Bleuler quantization[11]; infact we have a negative norm states: those generated by $\mu = 0$

[9] The equality $\frac{1}{2}\alpha_0^2 = -M^2\alpha'$ is due to the possibility to write p^μ in term of the Noether current corresponding to translations and to the relation $M^2 = -p^2$.

[10] Note that since the embedding functions are real we have $\alpha_n^\mu = (\alpha_{-n}^\mu)^*$ and similar for β modes.

[11] Gupta-Bleuler quantization is a way to quantize electromagnetic field. In this kind of quantization Lorentz invariance is kept manifest with inconvenience to have negative norm states which must be eliminated.

oscillation mode. These negative norm states must be eliminated and their elimination, we will see, strongly constrain dimension of background space. The vacuum of the theory is defined, as usual, as the state annihilated by all the annihilation operators and the excited states are generated applying creation operators on the vacuum. In the following we will indicate these states with $|\phi\rangle$.
Since oscillation modes are now operators also the objects defined in 2.19 become operators. It is not trivial the way L_m and $\tilde{L}_m$ become operators; we have products of operators and hence we have an ordering problem: we must consider normal ordering as usual in QFT

$$\hat{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\hat{\alpha}_{m-n}\cdot\hat{\alpha}_n:, \; \hat{\tilde{L}}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\hat{\beta}_{m-n}\cdot\hat{\beta}_n:; \tag{2.26}$$

these operators, called Virasoro operators, satisfy a generalization of Witt algebra called Virasoro algebra[12]

$$\begin{aligned} [\hat{L}_m, \hat{L}_n] &= (m-n)\hat{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}, \\ [\hat{\tilde{L}}_m, \hat{\tilde{L}}_n] &= (m-n)\hat{\tilde{L}}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}, \end{aligned} \tag{2.27}$$

where c is a central charge. The very interesting fact is that c is exactly the dimension D of the background space[13].
Classically we must have $L_m = \tilde{L}_m = 0\ \forall m$, but at the quantum level this cannot be true. First of all, the Virasoro operators have ordering problem; in details the operators $\hat{L}_0$ and $\hat{\tilde{L}}_0$ are the only two that have real problem in this sense. Hence, due to normal ordering, we cannot impose that they are trivially realized but we must impose

$$(\hat{L}_0 - a)|\phi\rangle = 0, \; (\hat{\tilde{L}}_0 - a)|\phi\rangle = 0; \tag{2.28}$$

the conditions 2.28 are called mass-shell conditions for the closed string[14], in case of open string we have only the first. Moreover, if we impose $\hat{L}_m|\phi\rangle = 0\ \forall m \neq 0$ we would have for $m, n \neq 0$

$$(\hat{L}_m\hat{L}_n - \hat{L}_n\hat{L}_m)|\phi\rangle = [\hat{L}_m, \hat{L}_n]|\phi\rangle = 0 \tag{2.29}$$

but, using Virasoro algebra, 2.27, and supposing $m + n \neq 0$ we get

$$(m-n)\hat{L}_{m+n}|\phi\rangle + \frac{c}{12}m(m^2-1)\delta_{m,-n}|\phi\rangle = 0; \tag{2.30}$$

this is true only if $c = 0$ or if $m = 0, \pm 1$. The first possibility is absurd, since in this case we would have a zero dimensional background space; the second one would restrict Virasoro algebra at only three elements. Hence, the best thing to do, is to impose only that

$$\hat{L}_{m>0}|\phi\rangle = 0. \tag{2.31}$$

[12] Formally, Virasoro algebra is the unique central extention of Witt algebra.
[13] See [40] pag. 80-81.
[14] For the closed strings we also learn, subtracting the two equations 2.28, that $\hat{L}_0 = \hat{\tilde{L}}_0$.

Identical analysis holds for the tilded Virasoro operators.
It is now the moment to talk about negative norm states. Between the states $|\phi\rangle$ there are those with negative norm; we can eliminate them constraining the Virasoro central charge c and the constant a and hence shrinking the accessible Fock space of the theory. The calculation is long[15] and the final result is very counterintuitive

$$a = 1, \;\; c = D = 26; \tag{2.32}$$

hence, to eliminate negative norm states must have $a = 1$ and we need a twentysix dimensional background space-time.[16] This is called critical string theory.
What we have done in canonical quantization it is possible to do in other ways. For example, we could have proceeded in the so-called light-cone quantization. In this type of quantization procedure we lose the Lorentz symmetry in favor of having only positive norm states, however we can force the Lorentz invariance by constraining a and c to have, respectively, the values 1 and 26.

Mass spectrum of bosonic string theory

At the quantum level, the mass formulas 2.21 and 2.22 undergo corrections due to normal ordering procedure; the modified equations are

$$\begin{aligned} &\textit{Open strings} \Rightarrow \hat{M}^2 = \frac{\hat{N} - 1}{\alpha'}, \\ &\textit{Closed strings} \Rightarrow \hat{M}^2 = \frac{4(\hat{N} - 1)}{\alpha'} = \frac{4(\hat{\tilde{N}} - 1)}{\alpha'}; \end{aligned} \tag{2.33}$$

where we have defined the number operators $\hat{N} := \sum_{n=1}^{\infty} : \hat{\alpha}_{-n} \cdot \hat{\alpha}_n :$ and $\hat{\tilde{N}} := \sum_{n=1}^{\infty} : \hat{\beta}_{-n} \cdot \hat{\beta}_n :$. Mass spectrum of the theory is classified according to the spectrum of number operators. Note that in case of closed strings the left and right oscillation modes are completely independent but that, for any given state, must be satisfied the level matching condition: $\hat{N} = \hat{\tilde{N}}$. At this point we are ready to study what kind of particles arise from bosonic strings oscillations. The mass spectrum is better understood in the light-cone quantization procedure, in which the only allowed oscillation mode are those transverse to the null coordinates; we will indicate them with the index $i = 1, ..., 24$.
For an open string we have:

- $N = 0$: we have only $|0\rangle$ with $M^2 = -\frac{1}{\alpha'}$. This is one scalar state and is a tachyon;
- $N = 1$: we have $\hat{\alpha}^i_{-1}|0\rangle$ with mass $M^2 = 0$. It has 24 states and it is massless vector boson belonging to the vector representation of $SO(24)$;
- $N = 2$: there are two possibilities $\hat{\alpha}^i_{-2}|0\rangle$ and $\hat{\alpha}^i_{-1}\hat{\alpha}^j_{-1}|0\rangle$ with mass $M^2 = \frac{1}{\alpha'}$. These have, respectively, 24 and 300 states and they combine to form the symmetric traceless second rank tensor representation of $SO(25)$.

[15] See [38] pag.65-70 for details.
[16] Note that this is true for a flat background space-time, condition that we have assumed in Polyakov action 2.5. In a generic background space-time it is possible to eliminate negative norm states also with $c = D \leq 26$ that are non critical theories.

- $N=3$: we have $\hat{\alpha}^i_{-3}|0\rangle$, $\hat{\alpha}^i_{-2}\hat{\alpha}^j_{-1}|0\rangle$ and $\hat{\alpha}^i_{-1}\hat{\alpha}^j_{-1}\hat{\alpha}^k_{-1}|0\rangle$ with mass $M^2=\frac{2}{\alpha'}$. So we have, respectively, 24, 576 and 2600 states. These combine to form some representations of $SO(25)$.

For a closed string we must take into account the level matching condition and the fact we have left and right movers. Moreover, the mass spectrum of a closed string can be deduced from that of the open string since a closed string state is a tensor product of left and right states, each of which has the same structure as open string states:

- $N=\tilde{N}=0$: we have only $|0\rangle$ with $M^2=-\frac{4}{\alpha'}$. This is one scalar state and is again a tachyon;
- $N=\tilde{N}=1$: we have $\hat{\alpha}^i_{-1}\hat{\beta}^j_{-1}|0\rangle$ with mass $M^2=0$. It has $24\times24=576$ states corresponding to the tensor product of two massless vectors. The symmetric traceless part transforms under $SO(24)$ as a massless spin two particle: the graviton $g_{\mu\nu}$. The trace is a massless scalar called dilaton, ϕ, while the antismmetric part $B_{\mu\nu}$, the so-called Kalb-Ramond (KR) field, transforms under $SO(24)$ as an antisymmetric tensor of rank two.

We see that the states belong to some representation of $SO(24)$ if are massless or $SO(25)$ if are massive; these two are the little groups as well as $SO(2)$ and $SO(3)$ are the little groups of four dimensional QFT. Moreover, there is a serious problem: due to the tachyonic state the vacuum is unstable.

2.1.3 Low energy behavior of bosonic string theory

Let us take a look at the low energy behavior of the theory. We note that α' controls the mass spectrum scale, so a low energy expansion is an expansion in $\alpha'\to0$. In this limit the massive excitations decouple and we are left with only massless modes. We might ask ourselves what happens if a string couples the massless modes of a closed string. The action that describes this coupling is

$$S=\frac{1}{4\pi\alpha'}\int d\tau d\sigma\sqrt{-h}\,\overbrace{[g_{\mu\nu}(\partial_\alpha X^\mu)(\partial_\beta X^\nu)h^{\alpha\beta}}^{coupling\ to\ graviton\ field}+\overbrace{iB_{\mu\nu}(\partial_\alpha X^\mu)(\partial_\beta X^\nu)\epsilon^{\alpha\beta}}^{coupling\ to\ KR\ field}+ \underbrace{+\alpha'\phi R^{(2)}]}_{coupling\ to\ dilaton\ field} \tag{2.34}$$

where $R^{(2)}$ is the Ricci scalar of the worldsheet[17]. We can think the KB field as a gauge potential with two indexes and so the string is electrically charged with respect to the KR field; we can also define a 3-form field strength for this field $dB_{\mu\nu}=dB_{(2)}=H_{(3)}=H_{\mu\nu\rho}=\partial_\mu B_{\nu\rho}+\partial_\nu B_{\rho\mu}+\partial_\rho B_{\mu\nu}$. Note that if the dilaton field is constant, the coupling with it is topological and comes through the Euler characteristic; this suggests that the constant mode of the dilaton, $\langle\phi\rangle$, determines

[17]In the following we will indicate with the apex (2) the Ricci tensor and scalar of the worldsheet while without the apex we will indicate the Ricci tensor and scalar of the background space.

the string coupling[18]. Nevertheless, there is a problem: the coupling to the dilaton field does not respect Weyl invariance unless the dilaton field is a constant. To force Weyl invariance we must impose that this lack is compensated by the one loop contributions arising from the coupling to graviton and KR fields. This can be done looking at the trace of the stress energy tensor that turn out to be

$$\langle T^\alpha_\alpha \rangle = -\frac{1}{2\alpha'}\beta^{(g)}_{\mu\nu} h^{\alpha\beta}(\partial_\alpha X^\mu)(\partial_\beta X^\nu) - \frac{i}{2\alpha'}\beta^{(B)}_{\mu\nu}\epsilon^{\alpha\beta}(\partial_\alpha X^\mu)(\partial_\beta X^\nu) - \frac{1}{2}\beta^{(\phi)}R^{(2)} \tag{2.35}$$

where

$$\begin{aligned}\beta^{(g)}_{\mu\nu} &= \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\phi - \frac{\alpha'}{4}H_{\mu\lambda\rho}H^{\lambda\rho}_\nu, \\ \beta^{(B)}_{\mu\nu} &= -\frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha'\nabla^\lambda\phi H_{\lambda\mu\nu}, \\ \beta^{(\phi)} &= -\frac{\alpha'}{2}\nabla^\mu\nabla_\mu\phi + \alpha'\nabla_\mu\phi\nabla_\mu\phi - \frac{\alpha'}{24}H_{\lambda\mu\nu}H^{\lambda\mu\nu}.\end{aligned} \tag{2.36}$$

Weyl invariance imposes traceless stress energy tensor and so $\beta^{(g)}_{\mu\nu} = \beta^{(B)}_{\mu\nu} = \beta^{(\phi)} = 0$; note that if the string is not coupled to the dilaton and KR fields, we get

$$\beta^{(g)}_{\mu\nu} = \alpha' R_{\mu\nu},\ \beta^{(B)}_{\mu\nu} \equiv 0,\ \beta^{(\phi)} \equiv 0, \tag{2.37}$$

and so the Weyl invariance request implies the Einstein's field equations in vacuum[19], $R_{\mu\nu} = 0$.

The set of equations $\beta^{(g)}_{\mu\nu} = \beta^{(B)}_{\mu\nu} = \beta^{(\phi)} = 0$ can be viewed as the equations of motion for the background in which the string propagates. We now change our perspective: we look for a $D = 26$ space-time action which reproduces these beta function equations as the equations of motion. This action is the low energy effective action of the bosonic string and it is

$$S = \frac{1}{16\pi G_{26}}\int d^{26}X\sqrt{-g}\Big(R + \frac{1}{6}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{e^{-\frac{\phi}{3}}}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big), \tag{2.38}$$

where G_{26} is the Newton's gravitational constant in twentysix dimensions[20]. Action 2.38 is nothing but Einstein gravity coupled with a scalar and a 2-form field. This is the reason why the field $g_{\mu\nu}$ is called graviton: string theory systematically contains gravity that emerges quite naturally in this context.

It is worth noting that we started with strings propagating in flat space-time and the quantum theory contains fluctuations making the space-time dynamical. In other words, the Poincaré symmetry, which is a global symmetry of the worldsheet action, becomes local from the background space-time theory point of view. This is a common feature of string theory: global symmetries on the worldsheet produce gauge symmetries in background space-time.

---

[18] This is an important point because means that the string coupling g_s that controls the genus expansion is not an independent parameter but it is defined by the theory itself.

[19] Note, however, that at more than one loop, α' corrections appear.

[20] It is important to note that the Newton's gravitational constants depends on the number of space dimensions. Thanks to dimensional arguments we find [41] $G_D = \frac{c^3 l^{D-2}_{P,D}}{\hbar}$ where $l^{D-2}_{P,D}$ is the Planck length in D dimensions.

2.2 Superstring theory

It is time to incorporate fermions in string theory: this is done thanks to SUSY and we talk about superstring. In reality the question is a little more subtle: the inclusion of fermions in string theory turns out to require SUSY.
To incorporate supersymmetry into string theory two basic approaches have been developed: the Ramond-Neveu-Schwarz (RNS) formalism and the Green-Schwarz (GS) formalism. The first one considers supersymmetry on the worldsheet while the second one considers it in the background space-time. It is intresting that in Minkowski space-time these two approaches are equivalent; in the following we will consider flat background space-time and we will use RNS formalism.
We start adding D fermionic fields[21] $\Psi^\mu(\tau,\sigma)$ to our D dimensional bosonic string theory; The fields $\Psi^\mu(\tau,\sigma)$ are two component spinors which describe fermions on the worldsheet and also trasform as vectors under Lorentz trasformations of the flat background space-time. Polyakov action is then modified to

$$S = \underbrace{-\frac{1}{4\pi\alpha'}\int d\tau d\sigma (\partial_\alpha X^\mu)(\partial^\alpha X^\mu)}_{bosonic\ part} \underbrace{-\frac{i}{4\pi\alpha'}\int d\tau d\sigma \bar{\Psi}^\mu \Gamma^\alpha \partial_\alpha \Psi_\mu}_{fermionic\ part}, \tag{2.39}$$

where the bar indicates Dirac adjoint and Γ^α are the two dimensional Dirac matrices

$$\Gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \tag{2.40}$$

since these matrices have real components, we are talking about Majorana spinors which have real components. It is possible to see[22] that action 2.39 has a global symmetry given by

$$\delta X^\mu = \bar{\epsilon}\Psi^\mu,\ \delta\Psi^\mu = \Gamma^\alpha \partial_\alpha X^\mu \epsilon \tag{2.41}$$

where ϵ is an infinitesimal Majorana spinor parameter. These transformations link boson fields to fermionic ones and viceversa, which lead exactly to supersymmetry and hence 2.39 is SUSY invariant action.

2.2.1 Boundary conditions and mode expansion

Using matrices 2.40 and 2.39 we can derive the equations of motion in the light-cone coordinates

$$\partial_+\partial_- X^\mu = 0,\ \partial_-\psi^\mu_+ = 0,\ \partial_+\psi^\mu_- = 0 \tag{2.42}$$

where $\psi^\mu_\pm$ are, respectively, the right and left component of the spinor Ψ^μ. Note that ψ^μ_- depends only to σ_- while ψ^μ_+ depends only to σ_+. Together with these equations we have to impose boundary conditions in order to vanish boundary terms present in the action's variation. For the bosonic part there are no news compared to before. For the fermion part we must impose contrains that led us to the so-called Ramond

[21] An essential request of SUSY is that the number of fermionic degrees of freedom are equal to the bosonic ones; this is why we add exactly D fermionic fields: to pair them with the D bosonic ones, $X^\mu(\tau,\sigma)$.

[22] For the full calculation see [38] pag. 165-167.

and Neveu-Schwarz sectors; moreover we have to distinguish the case of open and closed strings:

- Ramond sector for open strings: we impose $\psi_+^\mu(\tau,\sigma)\big|_{\sigma=0} = \psi_-^\mu(\tau,\sigma)\big|_{\sigma=0}$ and $\psi_+^\mu(\tau,\sigma)\big|_{\sigma=\pi} = \psi_-^\mu(\tau,\sigma)\big|_{\sigma=\pi}$. The fields admit the mode expansion

$$\psi_-^\mu(\sigma_-) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in\sigma_-},\ \psi_+^\mu(\sigma_+) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in\sigma_+}; \tag{2.43}$$

- Neveu-Schwarz sector for open strings: we impose $\psi_+^\mu(\tau,\sigma)\big|_{\sigma=0} = \psi_-^\mu(\tau,\sigma)\big|_{\sigma=0}$ and $\psi_+^\mu(\tau,\sigma)\big|_{\sigma=\pi} = -\psi_-^\mu(\tau,\sigma)\big|_{\sigma=\pi}$. In this case the fields admit the mode expansion

$$\psi_-^\mu(\sigma_-) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir\sigma_-},\ \psi_+^\mu(\sigma_+) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir\sigma_+}; \tag{2.44}$$

- Ramond sector for closed strings: in the case of closed strings we must impose different conditions, namely $\psi_\pm(\tau,\sigma) = +\psi_\pm(\tau,\sigma+\pi)$. The mode expansion is

$$\psi_-^\mu(\sigma_-) = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in\sigma_-},\ \psi_+^\mu(\sigma_+) = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in\sigma_+}; \tag{2.45}$$

- Neveu-Schwarz sector for closed strings: for this case we have antiperiodicity $\psi_\pm(\tau,\sigma) = -\psi_\pm(\tau,\sigma+\pi)$ and the mode expansion is

$$\psi_-^\mu(\sigma_-) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-2ir\sigma_-},\ \psi_+^\mu(\sigma_+) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{b}_r^\mu e^{-2ir\sigma_+}. \tag{2.46}$$

Note that we are using the convention according to which we use n or m for integer valued numbers and r or s for half integer ones. There is an important caveat to underline: while for open string we have only two sectors, Ramond (R) and Neveu-Schwarz (NS), in the case of closed strings we have four sectors. Left and right movers are independent and since a true closed strings state is the tensor product of right and left movers, we can construct R-R sector, R-NS sector, NS-R sector and NS-NS sector.

As for bosonic string theory, we must impose the worldsheet metric equations of motion. Writing them in light-cone coordinates we get the superstring theory analogue of Virasoro constrains

$$T_{++} = \partial_+X^\mu\partial_+X^\mu + \frac{i}{2}\psi_+^\mu\partial_+\psi_{+\mu} = 0,\ T_{--} = \partial_-X^\mu\partial_-X^\mu + \frac{i}{2}\psi_-^\mu\partial_-\psi_{-\mu} = 0. \tag{2.47}$$

2.2.2 Quantization of superstrings

Now is time to quantize the theory. The set of non vanishing commutation and anticommutation relations is[23]

$$\begin{aligned}
&[\hat{\alpha}^{\mu}_m, \hat{\alpha}^{\nu}_n] = [\hat{\beta}^{\mu}_m, \hat{\beta}^{\nu}_n] = m\eta^{\mu\nu}\delta_{m,-n};\\
&\{\hat{b}^{\mu}_r, \hat{b}^{\mu}_s\} = \{\hat{\tilde{b}}^{\mu}_r, \hat{\tilde{b}}^{\mu}_s\} = \eta^{\mu\nu}\delta_{r,-s};\\
&\{\hat{d}^{\mu}_m, \hat{d}^{\mu}_n\} = \{\hat{\tilde{d}}^{\mu}_m, \hat{\tilde{d}}^{\mu}_n\} = \eta^{\mu\nu}\delta_{m,-n}.
\end{aligned} \tag{2.48}$$

From 2.48 is easy to see that there are negative norm states; we know however, that with constrains on the dimension of background flat space-time and on the constants that emerge due to ordering problems, we can shrink the accessible Fock space and remove negative norm states. We talk about "constants" because we have more than one, in particular we have a constant for the R sector, a_R, and one for NS sector a_{NS}. The results of calculation are

$$a_R = 0,\ a_{NS} = \frac{1}{2},\ D = 10; \tag{2.49}$$

hence superstring theory in flat space-time has no negative norm states if the background space-time is ten dimensional. As we already know, the same result must be obtained if we use light-cone quantization but in this case we must impose 2.49 to recover Lorentz invariance of the theory. Moreover, in light-cone quantization only transverse oscillation modes are allowed; using this modes we can construct the mass spectrum for superstring. Before doing that, we must rattle off an important issue: the nature of R sector and NS sector ground states:

- NS sector ground state $|0\rangle_{NS}$: the ground state is unique and it corresponds to a bosonic state with spin 0; furthermore, since the oscillation modes are space-time vectors from the point of view of the background space-time, all excited states created from NS vacuum are space-time bosons;
- R sector ground state $|0\rangle_R$: this case is more tricky. Turn out that R vacuum is a set of degenerate spinorial states. Due to the fact that oscillation modes are space-time vectors follows that every states created form R vacuums are space-time fermions.

Mass spectrum of superstring theory

We start with open strings: the mass formulas for the two sectors are

$$\begin{aligned}
&NS\ sector \Rightarrow \hat{M}^2 = \frac{\hat{N}_{NS} - \frac{1}{2}}{\alpha'};\\
&R\ sector \Rightarrow \hat{M}^2 = \frac{\hat{N}_R}{\alpha'},
\end{aligned} \tag{2.50}$$

where $\hat{N}_{NS} := \sum_{n=1}^{\infty} : \hat{\alpha}_{-n} \cdot \hat{\alpha}_n : + \sum_{r=\frac{1}{2}}^{\infty} : r\hat{b}_{-r} \cdot \hat{b}_r :$ and $\hat{N}_R := \sum_{n=1}^{\infty} : \hat{\alpha}_{-n} \cdot \hat{\alpha}_n : + \sum_{n=1}^{\infty} : n\hat{d}_{-n} \cdot \hat{d}_n :$ are the number operators for the two sectors. Using 2.50 and the eight transverse mode labelled with the index $i = 1, ..., 8$, we have:

[23] As for the case of bosonic string theory, the embedding functions, bosonic or spinorial, are real hence $\alpha^{\mu}_n = (\alpha^{\mu}_{-n})^*$ and similar for all other modes.

- NS sector: the ground state $|0\rangle_{NS}$ has negative square mass and hence is a tachyon, this is a bad news. The first excited state is $\hat{b}^i_{-\frac{1}{2}}|0\rangle_{NS}$ and it has zero mass so it is a vector boson; its little group is $SO(8)$. The second excited states are $\hat{\alpha}^i_{-1}|0\rangle_{NS}$ and $\hat{b}^i_{-\frac{1}{2}}\hat{b}^j_{-\frac{1}{2}}|0\rangle_{NS}$ with mass $M^2 = \frac{1}{2\alpha'}$, they are 8 and 36 states that combine to form the symmetric traceless second rank tensor representation of $SO(9)$;

- R sector: the ground state $|0\rangle_R$ has zero mass and it is a 32 component spinor state[24]. However, in ten dimensions we must impose Majorana reality condition and Weyl condition; moreover it must satisfy the Dirac equation, hence we have 8 independent degrees of freedom. The first excited state are $\hat{\alpha}^i_{-1}|0\rangle_R$ and $\hat{d}^i_{-1}|0\rangle_R$ and they have $M^2 = \frac{1}{\alpha'}$.

We have seen that the NS sector ground state is a tachyon, however, we can use a procedure to project out this tachyonic state: Glozzi-Scherk-Olive (GSO) projection [43]. In a nutshell, GSO projection consists in taking only states that have a positive G-parity where the G operator is defined in the following way for the two sectors

$$\begin{aligned} &NS\ sector \Rightarrow \hat{G} := (-1)^{\hat{F}_{NS}+1} := (-1)^{\sum_{r\in\mathbb{Z}+\frac{1}{2}} \hat{b}_{-r}\cdot\hat{b}_r+1} \\ &R\ sector \Rightarrow \hat{G} := \tilde{\gamma}^{11}(-1)^{\hat{F}_R} := \tilde{\gamma}^{11}(-1)^{\sum_{n\in\mathbb{Z}} \hat{d}_{-n}\cdot\hat{d}_n} \end{aligned} \tag{2.51}$$

$\tilde{\gamma}^{11}$ is the ten dimensional analog of standard γ^5 Dirac matrix. From the definition above is easy to see that GSO projection is realized in the NS sector if we take only states with an odd number of $\hat{b}$ oscillator excitations while in the R sector we can take state with either even or odd number of $\hat{d}$ oscillator excitations depending on the chirality of the vacuum. GSO projection resolves our problem simply because the tachyon state is no more allowed: the new vacuum of the NS sector is the state $\hat{b}^i_{-\frac{1}{2}}|0\rangle_{NS}$.

We now move to closed strings: their mass spectrum can be obtained taking tensor products of open strings states; in the end we have four sectors: R-R, R-NS, NS-R, NS-NS. Recall that the GSO projection for the R sector depends on the chirality of the vacuum and that, in the case of closed strings, we have left and right movers each of them with his own R sector, this implies that we can built up two different theories depending on whether the G-parity of the left and right moving R sectors ground states is the same or opposite.

[24]This is because in $2k$ dimensions a spinor has 2^k component.

IIA superstring theory

In this case we have opposite chirality and we label them $|+\rangle_R$ and $|-\rangle_R$. The massless states in type IIA superstring theory closed string spectrum are given by

$$\begin{aligned}
&|-\rangle_R \otimes |+\rangle_R;\\
&\hat{\tilde{b}}^i_{-\frac{1}{2}}|0\rangle_{NS} \otimes \hat{b}^i_{-\frac{1}{2}}|0\rangle_{NS};\\
&\hat{\tilde{b}}^i_{-\frac{1}{2}}|0\rangle_{NS} \otimes |+\rangle_R;\\
&|-\rangle_R \otimes \hat{b}^i_{-\frac{1}{2}}|0\rangle_{NS}.
\end{aligned} \tag{2.52}$$

The R-R sector contains bosons since we are taking the tensor product of two spinors. This sector contains a 1-form gauge field ($C_{(1)}$, 8 states) and a 3-form gauge field ($C_{(3)}$, 56 states); the NS-NS sector contains bosons in particular it contains a scalar (the dilaton ϕ, 1 state), an antisymmetric 2-form gauge field (the KR field $B_{(2)}$, 28 states) and a symmetric traceless 2-form (the graviton $g_{\mu\nu}$, 35 states). For NS-R and R-NS sectors the particles content is the same and they contain fermions: a $\frac{3}{2}$ fermion (the gravitino, 56 states) and a $\frac{1}{2}$ fermion (the dilatino, 8 states). Since the two R vacuums have different chirality, the fermions that emerge from NS-R sector have opposite chirality with respect to those that emerge form R-NS sector.

IIB superstring theory

In this case we have same chirality, so we have only $|+\rangle_R$. The massless states in type IIB superstring theory closed string spectrum are given by

$$\begin{aligned}
&|+\rangle_R \otimes |+\rangle_R;\\
&\hat{\tilde{b}}^i_{-\frac{1}{2}}|0\rangle_{NS} \otimes \hat{b}^i_{-\frac{1}{2}}|0\rangle_{NS};\\
&\hat{\tilde{b}}^i_{-\frac{1}{2}}|0\rangle_{NS} \otimes |+\rangle_R;\\
&|+\rangle_R \otimes \hat{b}^i_{-\frac{1}{2}}|0\rangle_{NS}.
\end{aligned} \tag{2.53}$$

The R-R sector contains again bosons. This sector contains a 0-form gauge field ($C_{(0)}$, 1 state), a 2-form gauge field ($C_{(2)}$, 28 states) and a 4-form gauge field ($C_{(4)}$, 35 states). The NS-NS is identical to the previous case while for NS-R and R-NS sectors the particles content is the same and it is identical to the previous case but now the fermions that emerge from NS-R sector have same chirality with respect to those that emerge form R-NS sector. For this last fact we are most interested in type IIB superstring theory, this is a chiral theory.

2.2.3 Low energy effective actions for superstring theories

We have now seen that both IIA and IIB superstring theory contain the dilaton, the KR field and the graviton; moreover they contain other massless field depending on what type we are looking at. It is quite natural think to split up in three pieces the low energy effective action

$$S = S_1 + S_2 + S_F; \tag{2.54}$$

S_1 is the analogue of the bosonic string theory low effective action

$$S_1 = \frac{1}{16\pi G_{10}} \int d^{10}X\sqrt{-g}\Big(R + \frac{1}{6}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{e^{-\frac{\phi}{3}}}{2}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big) \tag{2.55}$$

where G_{10} is the Newton's constant in ten dimensions and $H_{\mu\nu\rho} = H_{(3)}$ is the 3-form field strength of the KR field $B_{(2)}$. S_2 depends of what type of theory we are considering:

$$\begin{aligned} IIA \Rightarrow S_2 &= -\frac{1}{32\pi G_{10}} \int d^{10}X\Big[\sqrt{-g}\Big(|F_{(2)}|^2 + |\tilde{F}_{(4)}|^2\Big) + B_{(2)} \wedge F_{(4)} \wedge F_{(4)}\Big]; \\ IIB \Rightarrow S_2 &= -\frac{1}{32\pi G_{10}} \int d^{10}X\Big[\sqrt{-g}\Big(|F_{(1)}|^2 + |\tilde{F}_{(3)}|^2 + \frac{1}{2}|\tilde{F}_{(5)}|^2\Big) + \\ &\quad + C_{(4)} \wedge H_{(3)} \wedge F_{(3)}\Big]; \end{aligned} \tag{2.56}$$

where $\wedge$ stands for wedge product[25] and $F_{(2)} = dC_{(1)}$, $F_{(4)} = dC_{(3)}$, $\tilde{F}_{(4)} = F_{(4)} - C_{(1)} \wedge H_{(3)}$, $F_{(1)} = dC_0$, $F_{(3)} = dC_{(2)}$, $F_{(5)} = dC_4$, $\tilde{F}_3 = F_{(3)} - C_{(0)} \wedge H_{(3)}$ and $\tilde{F}_{(5)} = F_{(5)} - \frac{1}{2}C_{(2)} \wedge H_{(3)} + \frac{1}{2}B_{(2)} \wedge F_{(3)}$. For type IIB theory we must also impose self Hodge duality[26] condition $\tilde{F}_{(5)} = *\tilde{F}_{(5)}$. The last term in 2.54 is S_F and describes the space-time fermionic interactions but we will not bother with here.

In the end, turn out that the low effective action for IIA and IIB superstring theory is $\mathcal{N} = 2$ supergravity: infact it is possible to show that the full action of superstring type II theories is invariant under $\mathcal{N} = 2$ local SUSY.

2.3 T-duality and Dp-branes

In this section we will study T-duality and we will see that the low effective action of N coincident Dp-branes is a $U(N)$ gauge theory. This fact is a crucial point for AdS/CFT correspondence.

We start with bosonic string theory but this time we consider one compactified dimension; our background space-time is $\mathbb{M}^{1,24} \times S^1_R$ so it is the product between twentyfive dimensional Minkowsky space and a circle of radius R. Let us consider

[25]Let $\mathcal{M}$ a real manifold of dimension D and consider its cotangent bundle $T^*\mathcal{M}$, this is the space of tensor fields of type $(0, 1)$. Consider now the so-called k-th exterior power of the cotangent bundle, $\bigwedge^k T^*\mathcal{M}$, this is the space of totally antisymmetric tensor fields of the type $(0, k)$, these are the k-forms. The wedge product is nothing but the product for this space, it is antisymmetric, associative and anticommutative. Wedge product between an r-form and an l-form is an $(r+l)$-form.

[26]Hodge star operator for a real manifold $\mathcal{M}$ of dimension D is an isomorphism of vector bundles $* : \bigwedge^k T^*\mathcal{M} \to \bigwedge^{D-k} T^*\mathcal{M}$ that associates to a k-form a unique $D - k$-form. Hodge star operator can be defined in a similar way also for complex manifolds. Recall that a vector bundle is a structure with three vector spaces and a map, (F, B, T, π), where F is the fiber, B is the base, T is the total space and π is a map between the total space and the base. Moreover we require that, for every $x \in B$ there is an open neighborhood $U \in B$ of x such that there is a homeomorfism $\phi : \pi^{-1}(U) \to U \times F$ in such a way that π commutes with the projection map onto the first factor of the space $U \times F$. Since the preimage of π is in E, this means that locally π is a projection map between E and the product space $B \times F$; hence, more intuitively the total space is locally a product space: $E \simeq B \times F$. A fiber bundle is said trivial if $E = B \times F$ globally.

closed strings, if we choose to compactify the 25th coordinate we must change the boundary condition on the 25th coordinate

$$X^{25}(\tau, \sigma = \pi) = X^{25}(\tau, \sigma = 0) + 2\pi RW \tag{2.57}$$

where $W \in \mathbb{Z}$ is the winding number and it counts the number of windings of the string around the compactified dimension. Moreover, also the mode expansion must be changed for the 25th embedding function and the most interesting news is that the 25th momentum coordinate must be discretized $p^{25} = \frac{K}{R}$ where K is called Kaluza-Klein (KK) exitation number. Hence, without compactified dimensions the center of mass momentum of the string is continuous while it becomes discreet along the compactified dimensions. This has an important implication on the mass formula: the idea is to interpret the mass formula from the point of view of the twentyfive dimensional theory in which the KK excitations, which are given by K, are considered as different particles. Starting from the mass-momentum relation in twentyfive dimensions we get the mass formula of a closed string with one compactified dimension

$$\hat{M}^2 = \left(\frac{K}{R}\right)^2 + \left(\frac{WR}{\alpha'}\right)^2 + \frac{2\hat{N} + 2\hat{\hat{N}} - 4}{\alpha'}; \tag{2.58}$$

we note that this mass formula is invariant under the transformation $R \to \frac{\alpha'}{R} := \tilde{R}$ as long as we interchange W and K; this symmetry is called T-duality of bosonic string. This is interesting; T-duality maps two bosonic string theories: one with compactified dimension of radius R and the other one with compactified dimension of radius $\tilde{R}$, into each other; note also that the interchange of winding number and KK number implies that momentum excitations, labelled by K in one description, correspond to winding mode excitations, labelled by W, in the dual description.

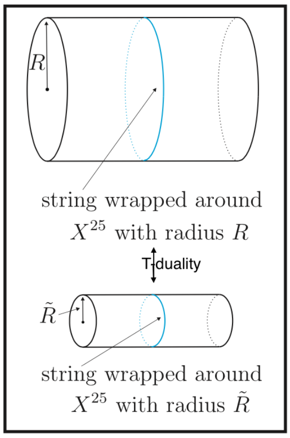

Figure 2.5. *Cartoon representing T-duality action, a theory compatified on a circle of radius R is mapped into a theory compactified on a cirle of radius $\tilde{R} = \frac{\alpha'}{R}$. Figure taken and modified from [39].*

Something even more interesting happens when we apply T-duality to a bosonic string theory with open strings: it turns out that T-duality transformation maps the compactified coordinate with NBC in a coordinate with DBC and viceversa. The endpoints of the dual open string are attached to an hyperplane, hence Dp-branes emerge naturally as objects in a T-dual theory. In general one can have more than one compactified dimension. If we start with a background space-time of the form $\mathbb{M}^{1,24-n} \times \mathbb{T}^n$, where $\mathbb{T}^n$ is the n-dimensionl torus[27], and if we apply T-duality to an open strings theory we will obtain a D$(25-n)$-brane. Note that if we take the particular limit $R \to 0$ this theory is physically equivalent to the one with $\tilde{R} \to \infty$. Consider now a configuration for our bosonic string theory with $\rho = 0, ...p$, NBCs and $\mu = p+1, ..., 25$ DBCs and so this theory has several Dp-branes and let us label with i and j two different Dp-branes. Let us consider an open string attached to different Dp-branes with coordinate x_i^μ and x_j^μ as in the following figure

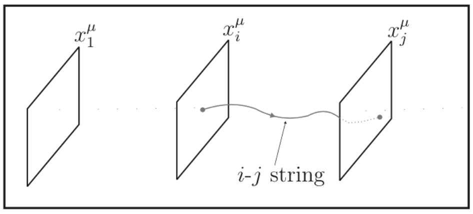

Figure 2.6. *This figure represents an open string attached with its endpoints at two different Dp-branes with coordinate x_i^μ and x_j^μ. We refer to this string as i-j string. Figure taken from [38].*

The mass formula for the i-j string is modified by the presence of the Dp-branes: the string will be more or less stretched and hence it has a tension T which contributes to the energy of the string,

$$\hat{M}^2 = \frac{\hat{N}-1}{\alpha'} + T^2(x_i^\mu - x_j^\mu)^2. \tag{2.59}$$

- One D$_p$-brane: $\hat{M}^2 = \frac{\hat{N}-1}{\alpha'}$ so only the states with $N=1$ are massless. This states are $\hat{\alpha}_{-1}^\rho|0\rangle$ or $\hat{\alpha}_{-1}^\mu|0\rangle$; the first corresponds to a $p+1$-dimensional vector A_ρ while the second to $25-p$ scalars X^μ with $\mu = 1, ...D-p-1$. Hence, we can interpret A_ρ as a gauge field living on the Dp-brane and the vacuum expectation values of X^μ as the position of the Dp-brane. This implies[28] that on the Dp-brane lives a $U(1)$ gauge theory;
- Two Dp-branes: $\hat{M}^2 = \frac{\hat{N}-1}{\alpha'} + T^2(x_i^\mu - x_j^\mu)^2$ with $i,j = 1,2$ and if we consider Dp-branes at the same position we have extra massless states with respect to the case with only one Dp-brane. We have the contributions of 1-2 string and 2-1 string since we are considering oriented strings[29]; similarly to before we get $(A_\rho)_{ij}$ and X_{ij}^μ, hence $(A_\rho)_{ij}$ is a gauge field of some $U(2)$ gauge theory living on the coincident Dp-branes;

[27] This can be defined as the product space of n circle $\mathbb{T}^n = S^1 \times ... \times S^1$. It is a n-dimensional manifold and it is not simply connected since its fundamental group is not trivial, $\pi_2 = \times^n \mathbb{Z}$. It can be seen also as fiber bundle, as an example $\mathbb{T}^2$ is a trivial S^1 bundle over S^1

[28] To see that, one have to consider the Dirac-Born-Infeld action with constant dilaton and vanishing KR field and must expand to lowest non trivial order in α'. For more details see [42] pag. 169.

[29] Roughly speaking, oriented strings are strings which can be thought having an internal "arrow"

- The generalization to N coincident Dp-branes is straightforward: we have $i, j = 1, .., N$ and so $(A_\rho)_{ij}$ and X^μ_{ij}; now the theory living on the N coincident Dp-branes is an $U(N)$ gauge theory[30] with gauge field $(A_\rho)_{ij}$.

To see what we have just listed in a rigorous way it would be necessary to introduce Chan–Paton factors; they are non-dynamical degrees of freedom from the worldsheet point of view, which are assigned to the endpoints of the string. These factors label the open strings that connect the various coincident Dp-branes. For example, the Chan–Paton factor λ_{ij} labels strings stretching from brane i to brane j, with $i, j = 1, ..., N$. The resulting matrix λ is an element of a Lie algebra and if strings are oriented it make sense to associate the fundamental representation, N, of this Lie algebra with the $\sigma = 0$ endpoint and the antifundamental representation, $\bar{N}$, with the $\sigma = \pi$ endpoint. It turns out that the only Lie algebra consistent with open string scattering amplitudes is $U(N)$ in the case of oriented strings, as we previously said. For strings that are unoriented the representations associated with the endpoints must be the same and this condition forces the symmetry group to be one with real fundamental representation: or $SO(N)$ or $USp(N)$.
There is yet another way to see that there exist a gauge theory living on N coincident Dp-branes: we know that an open strings theory in D dimensions contains a massles vector A_i with $i = 1, ..., D-1$; nevertheless the presence of a Dp-brane breaks the D-dimensional Poincaré symmetry, $ISO(D-1, 1)$, in the $p+1$-dimensional Poincaré symmetry of the Dp-brane and the $D-p-1$-dimensional rotational symmetry of the space trasverse at the Dp-brane, $ISO(p, 1) \times SO(D-p-1)$. From the D$p$-brane point of view, the D-dimensional gauge boson A_i produces a $p+1$ dimensional gauge boson A_ρ with $\rho = 0, ..., p$ and $D-p-1$ scalars $X^\mu = A_{p+i}$ with $i = 1, .., D-p-1$. This is exactly the same as before if we consider $D = 26$.
One may ask, what happens if one or more Dp-branes move away from the point of coincidence? The question would be very interesting and the answer is that the gauge group is spontaneous broken since the strings with the two endpoints on different Dp-branes can not have massless excitations, and the massive spectrum they produce is integrated out at low energy. The low energy effective field theory only contains the massless excitations arising from strings with both endpoints on coincident Dp-branes and so, for example, if the stack of N coincident Dp-branes is separated in two stacks, one made by N_1 coincident Dp-branes and one made by N_2 coincident Dp-branes we have the breaking pattern $U(N) \to U(N_1) \times U(N_2)$.

Dp-branes are present not only in toroidal compactified string theories: they really are states of the theory. However, this is not completely true for the bosonic string theory, and it is one of the many pathologies it displays. The reason is that quantum correction makes Dp-branes unstable. As usual, things are much more interesting for superstrings. First of all, what we have said for the bosonic strings remains true with obvious modifications, for example the dimensionality of the

which distinguishes the string from one with the opposite orientation. This means that a left to right direction on the string may be defined unambiguously. This is obvious since we parametrize the spatial extent by σ.

[30]Obviously this gauge theory is the low energy description since we are considering only massless states: we are describing the range of energy in which the massive states are not excited.

background space-time; moreover space-time supersymmetry plays a crucial role. Infact, by considering type IIA or IIB superstring theory with a closed string sector, by adding open strings ending on a Dp-brane 16 out of the 32 supercharges are preserved and for this reason Dp-branes are called $\frac{1}{2}$-BPS states. This implies that quantum corrections are milder, and Dp-branes might exist in the nonperturbative spectrum. This turns out to be true for certain p and their stability is guaranteed by the fact that they can carry conserved charges. The fundamental point is that as well as a particle couples to a 1-form, a Dl-brane couples to a $(l+1)$-form, hence a Dl-brane carries a charge with respect to this $(p+1)$-form. This charge is defined as

$$g_l^e = \int_{S^{8-l}} *F_{(l+2)} \tag{2.60}$$

and this is the electric charge of the Dl-brane and so we talk about electric Dl-brane; however there exist also magnetic brane, which, to avoid confusion, we label as Dq-brane, whose magnetic charge is defined by

$$g_q^m = \int_{S^{8-q}} F_{(q+2)}. \tag{2.61}$$

As we can see is not obvious that an electric and a magnetic brane have the same dimensionality; this happens if and only if $*F_{(l+2)}$ and $F_{(q+2)}$ are forms of the same rank[31]. S^{8-l} and S^{8-q} are the transverse space spheres surrounding the branes, while $*F_{(l+2)}$ and $F_{(q+2)}$ are the field strengths of the forms $C_{(l+1)}$ and $C_{(q+1)}$. In the following we will indicate electric brane as Dl-ebrane and magnetic one as Dq-mbrane In the end, with the knowledge of which forms populate the R-R sector of supestring theory we are considering, we are able to say which branes are stable based on the following steps:

$$\begin{array}{ccccccc} C_{(k)} & \xrightarrow{\textit{exterior derivative}} & F_{(k+1)} & \xleftrightarrow{\textit{Hodge dual}} & F_{(10-k-1)} & \xleftarrow{\textit{exterior derivative}} & C_{(10-k-2)} \\ \Big\downarrow \textit{gauge coupling} & & & & & & \textit{gauge coupling} \Big\downarrow \\ \mathrm{D}(k-1)-\textit{ebrane} & & & & & & \mathrm{D}(10-k-3)-\textit{mbrane} \end{array} \tag{2.62}$$

In type IIA superstring theory we have two forms: $C_{(1)}$ and $C_{(3)}$; so we get two electric branes and two magnetic branes. In the case of type IIB superstring theory there are three forms: $C_{(0)}$, $C_{(2)}$ and $C_{(4)}$ and hence there exist three electric branes and three magnetic ones. The content is summarized in Table 2.1.

We note an interesting fact: D3-brane is self dual; moreover, since the low energy effective field theories on Dp-branes are gauge theories and since a D3-branes is a four dimensional object and hence have a four dimensional worldvolume, we can think to somehow build up our ordinary four dimensional quantum field theories on D3-branes. This thought is at the base, as we will see in the next chapter, of the original AdS/CFT correspondence.

[31] This happens, for example, in ordinary electromagnetism in four dimensions where both the field strength $F_{(2)}$ and the dual field strength $*F_{(2)}$ are 2-forms.

	D_{-1}	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9
IIA	○	●	○	●	○	●	○	●	○	○	○
IIB	●	○	●	○	●	○	●	○	●	○	○

Table 2.1. *Electric and magnetic stable branes content in type II A and II B superstring theory. The full dot indicates the presence of stable brane in the theory while the empty dot indicate unstable branes and so its lack in the theory; electric branes are the first two while the last two are magnetic ones, this hold both in II A and II B type. D3-brane is very special: it is self dual.*

There is a final point to discuss: we now know that the low energy effective theory live on Dp-branes but previously we said that the low energy effective theory of suprestring theory is supergravity (we refer to this low energy effective limit supergravity as lowSUGRA), hence is reasonable to think that Dp-branes emerge from lowSUGRA [45]. Indeed it is the case and Dp-branes emerge as a classical solitonic solutions of the lowSUGRA equations of motion that generalize ordinary black holes[32]. They are BPS solutions and preserve half of the supercharges, it is a solution with symmetry $ISO(p,1) \times SO(9-p)$. Since we are most interested in type IIB superstring theory we consider only the Dp-brane solution of IIB lowSUGRA:

$$\begin{aligned} ds^2 &= H_p(r)^{-\frac{1}{2}}\eta_{\mu\nu}dx^\mu dx^\nu + H_p(r)^{\frac{1}{2}}dy^i dy^i, \\ e^\phi &= g_s H_p(r)^{\frac{3-p}{4}}, \\ C_{(p+1)} &= \left(H_p(r)^{-1} - 1\right)dx^0 \wedge dx^1 \wedge ... \wedge dx^p, \end{aligned} \tag{2.63}$$

where x^μ with $\mu = 0, ..., p$ are the coordinates on the Dp-brane worldvolume, y^i with $i = p+1, ..., 9$ are the coordinates perpendicular to the Dp-brane, $r := \sum_i y^i y_i$ and the function $H_p(r)$ have to be harmonic for $r \neq 0$. The function $H_p(r)$ turns out to be

$$H_p(r) = 1 + \frac{(4\pi)^{\frac{5-p}{2}}\Gamma\left(\frac{7-p}{2}\right)g_s N\alpha'^{\frac{7-p}{2}}}{r^{7-p}}, \tag{2.64}$$

where $\Gamma(\,\cdot\,)$ is the Euler gamma function and N the so-called units of R-R flux and is obtained from equation 2.60 for electric branes and from equation 2.61 for magnetic branes.

S-duality

In the context of string dualities there is yet another important duality: S-duality. While T-duality relates two different compactified theories, S-duality is a weak-strong coupling duality, that relates different regimes of the same theory. The most prominent example where S-duality is present is type IIB superstring theory; this theory is mapped to itself under S-duality. This is due to the fact that S-duality is a special case of the $SL(2,\mathbb{Z})$ symmetry of type IIB superstring theory. Arranging

[32]The idea is that if we consider a superstring theory compactified on a compact manifold down to 4 dimensions; hence a black hole is construct considering a configuration of intersecting wrapped in the compact manifold Dp-branes which upon dimensional reduction yields a black hole space-time.

the R-R 0-form $C_{(0)}$ and the dilaton ϕ in a complex scalar $\tau := C_{(0)} + ie^{-\phi}$, the $SL(2,\mathbb{Z})$ symmetry of the equations of motion of type IIB supergravity acts as

$$\tau \to \frac{a\tau + b}{c\tau + d}; \tag{2.65}$$

where a, b, c, d are integer parameters satisfying $ad - bc = 1$. In the special case $a = d = 0$ and $b = -c = 1$, we get

$$\tau \to -\frac{1}{\tau} \tag{2.66}$$

and if $C_{(0)}$ vanishes we obtain

$$\tau := ie^{-\phi} = \frac{i}{g_s} \to -\frac{1}{\tau} := -\frac{e^{\phi}}{i} = -\frac{g_s}{i} \Rightarrow \frac{1}{g_s} \to g_s \tag{2.67}$$

and this is a weak-strong duality called S-duality.

Chapter 3

Gauge/gravity duality

Gauge/gravity duality is one of the major new development within theoretical physics in the last twentyfive years; it was originally proposed by Juan Maldacena in 1997 [46], it brings together string theory, quantum field theory and gravity, and has applications to elementary particle and nuclear physics as well as condensed matter physics and hydrodynamics.

Gauge/gravity duality is of fundamental importance since it provides new interesting links between quantum theory and gravity which are based on string theory and, precisely, most on Dp-branes; the duality maps strongly coupled quantum field theories, which are generically hard to describe, to more tractable classical gravity theories that arise from string theory. The important point of gauge/gravity duality is that it realises the holographic principle and is therefore referred to as holography. Holographic principle [47],[48] states that the entire information content of a quantum gravity theory in a given volume can be encoded in an effective theory at the boundary surface of this volume; it is based on black holes thermodynamics. Looking at the first principle of black hole thermodynamics in the case of a discharge non rotating black hole

$$dM = \left(\frac{1}{8\pi M}\right)\frac{dA}{4} = T_H\frac{dA}{4} \tag{3.1}$$

and remembering that in thermodynamics $dE = TdS$, we get the entropy of a Schwarzschild black hole

$$S = \frac{A}{4}. \tag{3.2}$$

So the information of a $(D+1)$-dimensional system is encoded in its boundary D-dimensional area.

Until now, the most prominent and best understood example of gauge/gravity duality and of holographic principle is the celebrated Anti de Sitter/Conformal Field Theory (AdS/CFT) correspondence and, as we said, was proposed by Juan Maldacena in 1997 [46]. The AdS/CFT correspondence is characterized by a very high degree of symmetry; infact the field and gravity theories involved in the AdS/CFT correspondence display both supersymmetry and conformal symmetry; we will see that the theory of quantum gravity involved is defined on a manifold of the form AdS $\times S^5$ where S^5 is the five dimensional sphere. The quantum field theory may be thought of as being defined on the conformal boundary of this Anti de Sitter space-time that we will see to be Minkowsky space-time plus conformal symmetry. Briefly speaking, AdS/CFT correspondence states that $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ and Yang–Mills coupling constant g_{YM} is dynamically equivalent to a type IIB superstring theory with string lenght l_s and string coupling g_s on AdS $\times S^5$, with radius of curvature L and N units of $F_{(5)}$ R-R flux on S^5.

Motivated by the successes of the AdS/CFT correspondence in its original form, many physicists have begun to ask the question whether the AdS/CFT correspondence can be used to shed new light onto open problems in theoretical physics which are linked to strong coupling. However, although approaches to describing some of their properties exist, there is no general methods to calculate their observables. Consequently, new ideas for describing strongly coupled systems are very welcome, and generalisations of the AdS/CFT correspondence to more general gauge/gravity dualities have made useful contributions to new descriptions of at least some aspects of strongly coupled systems. The best established example is given by the combination of gauge/gravity duality methods with linear response theory, for describing transport processes; among others interesting phenomena of strong coupling, which have been investigated using gauge/gravity duality, there is the description of theories related to QCD at low energies and the applications to the physics of the quark–gluon plasma[1] [51]. More recently, gauge/gravity duality has also been applied to strongly coupled systems in condensed matter physics [49],[50].

In this chapter we will study the original AdS/CFT correspondence [42],[52],[53],[54] first giving some elements on the large N limit [57] of a gauge theory and on the Anti de Sitter space-time [55],[56]. In the last section we will start to see how to extend AdS/CFT correspondence to have more interesting theories from the pheonomenological point of view; we will see that instead of a high symmetric sphere the choice will fall on a Sasaki-Einstein space that we will introduce [58],[59].

3.1 Large N limit and strings

The first one to propose to consider a large N limit of a $SU(N)$ or $SO(N)$ Yang-Mills theory was Gerardus 't Hooft; infact, motivated by an expansion used in statistical mechanics, where the number of field components is taken to be large and an expansion in the inverse of this number is performed, 't Hooft proposed [57] that non abelian gauge theories simplify considerably in the limit of large N, $N \to \infty$. Consider a $SU(N)$ Yang-Mills theory without fermions and with coupling g, its one loop beta function is

$$\beta(g) = -\frac{11Ng^3}{48\pi^2}, \tag{3.3}$$

so we see that in the limit $N \to \infty$ the beta function diverges. However, if we consider $\lambda := g^2 N$ fixed while taking the large N limit then the renormalisation

[1]Quark-gluon plasma (QGP) is an interacting localized assembly of quarks and gluons at thermal and chemical equilibrium in which free color charges are allowed, infact in normal matter quarks are confined while in the QGP quarks are deconfined. The study of the QGP is also a testing ground for finite temperature field theory, a branch of theoretical physics which seeks to understand particle physics under conditions of high temperature. Such studies are important to understand the early evolution of our universe.

group equation for the 't Hooft coupling λ does not diverge at one loop,

$$\beta(\lambda) = \mu \frac{d\lambda}{d\mu} = 2Ng\mu \frac{dg}{d\mu} = -\frac{11N^2 g^4}{24\pi^2} = -\frac{11\lambda^2}{24\pi^2}; \tag{3.4}$$

so the large N limit with the 't Hooft coupling kept fixed, exists in a non trivial way since the corresponding field theory is not free as we can see from 3.4. We highlight that in this limit the effective coupling constant is not g but λ.
We restrict our interest to toy model scalar theory with gauge group $U(N)$ and fields $g\Phi^i_j \equiv g\Phi^a(T_a)^i_j$ in the fundamental representation,

$$\mathcal{L} = \frac{1}{g^2}\Big[-\frac{1}{2}Tr(\partial_\mu \Phi \partial^\mu \Phi) + Tr(\Phi^3) + Tr(\Phi)\Big]; \tag{3.5}$$

the free propagator for the scalar fields is[2]

$$\Big\langle \Phi^i_j(x)\Phi^k_l(y)\Big\rangle = \frac{g^2}{4\pi^2(x-y)^2}\delta^i_l \delta^k_j, \tag{3.6}$$

this expression for the propagator suggests a double line notation as shown in Figure 3.1.

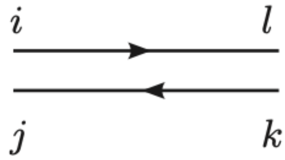

Figure 3.1. *Double line notation of a field; the arrow on each line points from an upper to a lower index. Figure taken from [42].*

Feynman diagrams then become networks of double lines. If we introduce the 't Hooft coupling we read from the lagrangian density 3.5 that the kinetic term scales as $g^{-2} = \frac{N}{\lambda}$ while the propagator, being the inverse of the kinetic term, scales as $g^2 = \frac{\lambda}{N}$; moreover the sum over indices in a trace contributes a factor N for each closed loop. If we introduce the shorthand notation (V, E, F) for the numbers of vertices, propagators (edges) and loops (faces) respectively, a Feynman diagram with V vertices, E propagators and F loops turns out to be proportional to

$$diagram(E, V, F) \propto N^{V-E+F}\lambda^{E-V} = N^{\chi}\lambda^{E-V}; \tag{3.7}$$

the very interesting fact is that the diagrams are proportional to the Euler characteristic, $\chi = V - E + F = 2 - 2h$, where h is the genus.
Physical quantity in this theory can be expressed in an expansion of N and h; for

[2] In a $SU(N)$ Yang-Mills theory there would be a term proportional to $\frac{\delta^i_j \delta^k_l}{N}$ which is suppressed in the large N limit. Hence for gauge group $SU(N)$ and $U(N)$ the structure of propagators is analogue and holds the same considerations.

example, the generating functional for connected Green's functions[3] can be written

$$ln(Z) = \sum_{h=0}^{\infty} N^{2-2h} P_h(\lambda), \tag{3.8}$$

where $P_h(\lambda)$ is a complicated polynomial in the 't Hooft coupling. From 3.8 it is evident how the generating functional can be thought as a topological perturbative expansion, in which "topological" is due to the fact that the expansion is organized in surfaces with fixed genus. For large N the series is dominated by surfaces of minimal genus $h = 0$, the so-called planar diagrams; the surfaces expansion emerges from the fact that the left diagram in Figure 3.2 can be drown on a sphere while there is no way to draw the right diagram on a sphere: we need a torus to draw it over.

Figure 3.2. *Examples of possible diagrams in exapnsion 3.8. The left diagram has* $E = 3, F = 3, V = 2$ *and so scales as* N^2*: it is a planar diagram. The right diagram has* $E = 6, F = 2, V = 4$ *and so scales as* N^0*: it is not a planar diagram. The right diagram would be suppressed in the limit of large* N*. Figure taken from [42].*

The crucial point of 3.8 is that the large N expansion is formally the same as a perturbation expansion of closed oriented strings with string coupling $\frac{1}{N}$; hence the behavior of a gauge theory with group $U(N)$ or $SU(N)$ is dynamically equivalent to a closed oriented string theory.
Two recommendations are in order now: first, for $SO(N)$ or $USp(N)$ theories, the adjoint representation may be written as a product of two fundamental representations rather than a product of a fundamental and an antifundamental representation-, since the fundamental representation is real, there are no arrows on the propagators and we expect the planar diagrams obtained to be associated to a non orientable string theory. Secondly, in general for these toy model theories it is not know which string theory fits the field theory perturbative series; however in the special case of $\mathcal{N} = 4$ Super Yang–Mills theory, the AdS/CFT correspondence tells us which string theory leads to the correct expansion: ten dimensional type IIB superstring theory on AdS $\times S^5$ background. Moreover, thanks to deformations of AdS/CFT correspondence we are able to find other examples where we can link the string perturbative expansion to the field theory one.

3.2 Anti de Sitter space-time

We saw that one important ingredient of AdS/CFT correspondence is Anti de Sitter space-time, so now it is time to talk about it.

[3] Recall that the partition function Z generates all Green's functions; its logarithm, $ln(Z)$, generates connected Green's functions and its Legendre transform with respect to the external sources, Γ, called quantum action, generates only the 1PI Green's functions.

First of all we have to introduce the concept of maximally symmetric space [60]. Let us consider a D-dimensional differentiable real manifold $\mathcal{M}$ with metric $g_{\mu\nu}$, and consider the Killing equation for a vector field V

$$L_V g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu = 0, \tag{3.9}$$

where L_V is the Lie derivative and ∇_α is the covariant derivative. The question is: how many linear independent Killing fields can a manifold have? It can be shown that a manifold of dimension D can have at most $\frac{D(D+1)}{2}$ linearly independent Killing vector fields; space-times which satisfy this bound, and hence have as many Killing fields as possible, are called maximally symmetric space-times. For this kind of space-times it is possible to express the Riemann tensor in terms of the Ricci scalar

$$R_{\mu\nu\rho\sigma} = \frac{R}{D(D-1)}(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma}). \tag{3.10}$$

Therefore we see that we can classify maximally symmetric space-times according to their dimension and the value of the Ricci scalar as long as the space-time manifold is lorentzian[4]; for $R = 0$, the maximally symmetric space-time is Minkowski space-time, for $R < 0$ the maximally symmetric space-time is Anti de Sitter and for $R > 0$ the maximally symmetric space-time is de Sitter. Moreover, it is easy to show that these three maximally symmetric space-times are related, in $D > 2$, to the presence, respectively, of a vanishing, negative and positive cosmological constant, Λ, in the Einstein's field equations in vacuum[5]. Hence, Anti de Sitter space-time is a maximally symmetric solution of Einstein's field equation in vacuum with negative cosmological constant.

In the following we will give some detail of AdS space-time. $(D+1)$-dimensional Anti de Sitter space-time, AdS$_{D+1}$, can be embedded into $(D+2)$-dimensional Minkowski space-time with coordinates $(x^0, x^1, ..., x^{D+1})$ and metric $\bar{\eta} = diag(-1, +1, ..., +1, -1)$ as the hypersurface generated by the constrain

$$\bar{\eta}_{\alpha\beta}x^\alpha x^\beta = -\left(x^0\right)^2 + \sum_{i=1}^{D}\left(x^i\right)^2 - \left(x^{D-1}\right)^2 = -L^2, \tag{3.11}$$

with $\alpha, \beta = 0, ..., D-1$ and L plays the role of the radius; note that AdS$_{D+1}$ shows an isometry group $SO(D, 2)$. Let us study different coordinate systems for describing AdS$_{D+1}$; we are interested in three coordinates system which will give us interesting information:

1. Global coordinates: consider the parametrization of the hyperboloid

$$\begin{aligned} x^0 &= Lcosh(\rho)cos(\tau), \\ x^i &= L\Omega_i sinh(\rho) \quad 1, ..., D, \\ x^{D+1} &= Lcosh(\rho)sen(\tau), \end{aligned} \tag{3.12}$$

[4]Obviously it is possible to classify also riemannian space: we have the flat euclidean space ($R = 0$), the sphere ($R > 0$) and the hyperboloid ($R < 0$). However they are not of our interest momentarily.

[5]Taking the field equations in vacuum and contracting with $g^{\mu\nu}$ we obtain $g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}) = R - \frac{1}{2}DR + \Lambda D = 0$ and so $R = \frac{-\Lambda D}{1-\frac{1}{2}D} = \frac{2\Lambda D}{D-2}$.

where Ω_i with $i = 1, ..., D$ are angular coordinates satisfying $\sum_i \Omega_i^2 = 1$ that parametrizes a $(D-1)$-dimensional sphere; $\rho \in \mathbb{R}^+$ and $\tau \in [0, 2\pi)$. The coordinates system (ρ, τ, Ω_i) is called global system because it describes the whole AdS space-time. In this coordinates the metric is

$$ds^2 = L^2\big[d\rho^2 - cosh^2(\rho)d\tau^2 + sinh^2(\rho)d\Omega_{D-1}^2\big], \tag{3.13}$$

and it admits a timelike Killing vector associated to the coordinate τ, hence we call τ global time coordinate. However, τ is defined on a compact manifold with period 2π, so AdS space-time suffers from the presence of Closed Timelike Curve (CTC) as drown in the case of AdS_2 in Figure 3.3. To avoid inconsistencies[6], we should consider the universal covering of Anti de Sitter space-time by unwrapping the timelike circle taking $\tau \in \mathbb{R}$ without identifications. Hence, in the following when we talk about Anti de Sitter space-time we always refer to universal covering of AdS space-time.

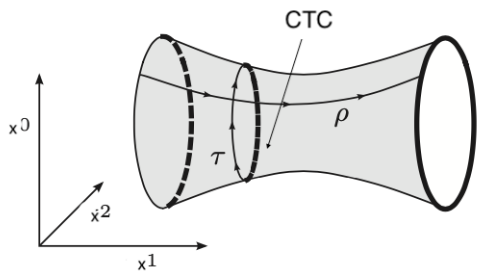

Figure 3.3. *Picture of AdS_2; we have emphasized the presence of a CTC. Figure taken and modified from [42]*

Note that the spatial section at constant ρ are D-dimensional spheres.

2. Einstein static universe coordinates: we introduce the new coordinate θ by the relation $tan(\theta) := sinh(\rho)$; with this definition inserted in metric 3.13 we get

$$ds^2 = \frac{L^2}{cos^2(\theta)}\big[-d\tau^2 + d\theta^2 + sin^2(\theta)d\Omega_{D-1}^2\big]; \tag{3.14}$$

this is the metric of a space with cylindrical geometry, $\mathbb{R} \times S^D$, spatial sections. Metric 3.14 also tell us something more important: Anti de Sitter space-time admits a boundary ∂AdS, called conformal boundary, located at $\theta = \frac{\pi}{2}$. Since the prefactor $\frac{L^2}{cos^2(\theta)}$ is never negative we can eliminate it with a conformal transformation; this implies that at the boundary the metric reduces to

$$ds^2_{\partial AdS} = \big[-d\tau^2 + d\Omega_{D-1}^2\big] \tag{3.15}$$

which is the metric of a D-dimensional compactified Minkowski space-time.

[6]CTC are violation of causality: if one travels along a CTC, going forward in time he would find himself in the past; this is obviously not physical. This type of geodesics was discovered by Kurt Gödel in 1949 [61] during his study on cosmological solution of Einstein's field equations.

3. Poincaré coordinates: consider the parametrization of the hyperboloid

$$
\begin{aligned}
x^0 &= \frac{L}{2r}\left[1 + \frac{L^2}{r^4}(\vec{y}^2 - t^2 + L^2)\right],\\
x^i &= \frac{ry^i}{L} \quad i = 1, ..., D-1,\\
x^D &= \frac{L}{2r}\left[1 + \frac{L^2}{r^4}(\vec{y}^2 - t^2 - L^2)\right],\\
x^{D+1} &= \frac{rt}{L},
\end{aligned} \tag{3.16}
$$

where $t \in \mathbb{R}$, $r \in \mathbb{R}^+$ and $\vec{y} := (y^1, ..., y^{D-1})$ is a set of coordinates of $D-1$-dimensional euclidean space. Since $r > 0$ we cover only half of the AdS space-time. These local coordinates are referred to as Poincaré patch coordinates. In the Poincaré patch, shown in Figure 3.4, the metric is

$$
ds^2 = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}\eta_{\mu\nu}dy^\mu dy^\nu; \tag{3.17}
$$

conformal bundary is located at $r \to \infty$. On other hands, at $r \to 0$, we have the so-called Poincaré horizon, that is a coordinate singularity[7].

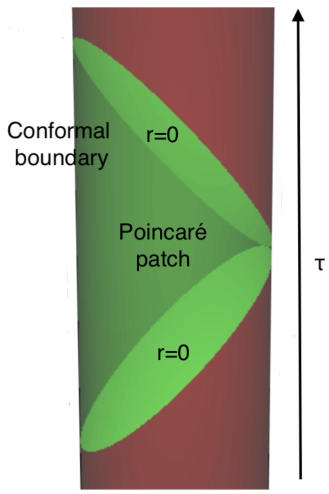

Figure 3.4. *Representation of AdS space-time, the area in green is the Poincaré patch described by the metric 3.17; this is only half of the whole space-time. On the left we have the conformal boundary while the two sections are the Poincaré horizons.*

[7]Remember that a coordinate singularity can be avoided with a suitable change of coordinates, infact we know that in the global coordinate there is no singularity. The concept of coordinate singularity is opposite to the concept of gravitational singularity where the curvature invariants diverge; the latter is a true singularity and no change of coordinates can safe you from it. Obviously, it is no really physical the fact that a region of space-time has infinite curvature and so infinite energy: there should be a more powerful or more appropriate theory that can describe what really happens.

We can write the metric 3.17 in another way: let us define $z := \frac{L^2}{r} \Rightarrow dz = -\frac{L^2}{r^2}dr$, then the metric becomes

$$ds^2 = \frac{L^2}{r^2}\frac{r^4}{L^4}dz^2 + \frac{r^2}{L^2}\eta_{\mu\nu}dy^\mu dy^\nu = \frac{L^2}{z^2}\left[dz^2 + \eta_{\mu\nu}dy^\mu dy^\nu\right]. \tag{3.18}$$

In this case the horizon is at $z \to \infty$ and the boundary is at $z \to 0$.

We know that the isometry group of AdS is $SO(D, 2)$, and now we have said that AdS has a boundary conformal to Minkowski space-time; hence we arrive to an interesting conclusion: AdS conformal boundary heritages isometries of AdS like conformal symmetry and since ∂AdS is conformal to Minkowski space-time we get Minkowski space-time with conformal symmetry as boundary of AdS space-time. This conclusion will be of enormous importance to arrive at AdS/CFT correspondence.

3.3 The AdS/CFT correspondence

We are now arrived to discuss AdS/CFT correspondence that is one of the most important concepts and tools of this work. In the first paragraph we will study how the corrispondence emerges from the framework of superstrings while in the second paragraph we will better understand how is built up the correspondence or duality map.

3.3.1 Emergence of the correspondence

We have learned that Dp-branes are two faces: high dimensional objects on which open strings can end and for which the low energy dynamics is described by a $U(N)$ gauge theory or solitonic solutions of the low energy limit of superstring theory, lowSUGRA, where they are considered as sources of gravitational field which curves the surrounding space-time. This two different but complementary point of view, when applied to the special case of a stack of N D3-branes allow us to motivate how AdS/CFT correspondence emerges. Let us study this two perspectives in detail. We refer to the first as open string perspective and to the second as closed string perspective:

- Open string perspective: in this case strings must be treated as small perturbations and so $g_s << 1$. We consider type IIB superstring theory in flat ten dimensional Minkowski spacetime where we also embed N coincident D3-branes, placed in the directions x^0, x^1, x^2, x^3, at low energy $E << \frac{1}{\sqrt{\alpha'}}$. In other words, we take only massless excitations into account and ignore all others since they are masses of order $\frac{1}{\sqrt{\alpha'}}$. Perturbative string theory in this background consists of two kinds of strings: open strings beginning and ending on the D_3-branes and closed strings; furthermore this setup preserve half of the SUSY charges of type IIB superstring theory, hence sixteen supercharges are preserved. Massless excitations can be grouped into supermultiplets; to be precise, the massless open string excitations[8], a four dimensional gauge boson

[8] Remember that this is the spectrum from the point of view of the D3-branes. See pag. 53-54 for a refresh if needed.

A_ρ and six scalars X^μ with $\mu = 1, ..., 6$, may be grouped into a four dimensional $\mathcal{N} = 4$ supermultiplet which contains also their fermionic superpartners while the massless closed string excitations form a ten dimensional $\mathcal{N} = 1$ lowSUGRA multiplet. We also know that the dynamic of this low energy limit is described by a $U(N)$ gauge theory, so we will take traces over gauge indexes. The total action for this low energy effective theory can be written as the sum of three pieces

$$S = S_{closed} + S_{open} + S_{int}, \tag{3.19}$$

where S_{closed} is given by 2.55 while S_{open} and S_{int} turn out to be[9]

$$\begin{aligned} S_{open} = &-\frac{1}{2\pi g_s}\int d^4x\Big(\frac{1}{4}Tr(F_{\rho\sigma}F^{\rho\sigma}) + \frac{1}{2}\sum_{\mu=1}^{6}\nabla_\rho X^\mu \nabla^\rho X^\mu + \\ &- \sum_{\mu,\nu=1}^{6} Tr[X^\mu, X^\nu]^2\Big) + \mathcal{O}(\alpha'), \\ S_{int} = &-\frac{1}{8\pi g_s}\int d^4x\ c\phi Tr(F_{\rho\sigma}F^{\rho\sigma}), \end{aligned} \tag{3.20}$$

where $F_{\rho\sigma}$ with $\rho, \sigma = 0, ..., 3$ is the field strength of the gauge vector A_ρ living on the N coincident D_3-branes, ϕ is the dilaton and c is a constant of order α'. We now take the limit $\alpha' \to 0$: S_{int} vanishes and so open and closed strings decouple, S_{closed} turn out to be the action of free type IIB lowSUGRA in ten dimensional Minkowski space-time and S_{open} reduces to the action of a $D = 4$ $\mathcal{N} = 4$ SYM gauge theory with $U(N)$ gauge group provided that we identify

$$2\pi g_s = g_{YM}^2 \tag{3.21}$$

we will refer to this limit as decoupling limit.

- Closed string perspective: in this case we have a strong coupling $g_s \to \infty$, and the stack of N D3-branes may be viewed as massive charged objects sourcing various fields of type IIB lowSUGRA, and therefore also of type IIB string theory; this is the background where closed strings of type IIB superstring theory will propagate. We know that D_p-branes are soliton solutions preserving $ISO(p, 1) \times SO(D - p - 1)$ isometries of $\mathbb{M}^{9,1}$ and half of the supercharges of type IIB lowSUGRA. The metric of these solutions is given by 2.63 that in the case of D3-branes become

$$\begin{aligned} ds^2 &= H_3(r)^{-\frac{1}{2}}\eta_{\mu\nu}dx^\mu dx^\nu + H_3(r)^{\frac{1}{2}}dx^i dx^i, \\ e^\phi &= g_s, \\ C_{(4)} &= \big(H_3(r)^{-1} - 1\big)dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \end{aligned} \tag{3.22}$$

where x^μ with $\mu = 0, ..., 3$ are the coordinates on the D3-brane worldvolume, x^i with $i = 4, ..., 9$ are the coordinates perpendicular to the D3-brane, $r := \sum_i x^i x_i$ and the function $H_3(r)$ is, looking at 2.64

$$H_3(r) = 1 + \frac{4\pi g_s N\alpha'^2}{r^4} = 1 + \frac{L^4}{r^4}, \tag{3.23}$$

[9]They can be derived from the so-called Dirac-Born-Infeld action; see [42] pag. 185 for more details.

where we have defined the characteristic length $L^4 := 4\pi g_s N \alpha'^2$. This definition can be rewritten looking at 3.21 as

$$2g_{YM}^2 N = \frac{L^4}{\alpha'^2} \tag{3.24}$$

We have two regions in the background: for $r >> L$, corresponding to the asymptotic region, we have $H_3(r) \simeq 1$ and so metric 3.22 reduces to a flat minkowskian metric in ten dimension, while for $r << L$, corresponding to the near horizon region, we have $H_3(r) \simeq \frac{L^4}{r^4}$ and the metric 3.22 becomes

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dx^i dx^i, \tag{3.25}$$

which, introducing the spherical coordinates $(r, \Omega_5) \in \mathbb{R}^+ \times S^5$ and $z = \frac{L^2}{r} \Rightarrow dz = -\frac{L^2}{r^2}dr$, can be rewritten as

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}(dr^2 + r^2 ds^2_{S^5}) = \frac{L^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) + L^2 ds^2_{S^5}. \tag{3.26}$$

The second part of the metric is a five dimensional sphere while the first part, looking at metric 3.18, is exactly AdS_5 space-time. We can do this simple consistency check: when $z \to \infty$ and so $r \to 0$ we move towards the horizon of AdS_5 space-time and this explains why previously we called the limit $r << L$ near horizon region: we are close to the AdS_5 Poincaré horizon with the other five dimensions compactified in a sphere; on other hand, when $z \to 0$ and so $r \to \infty$ we move towards the conformal boundary of AdS space-time: a four dimensional Minkowski space-time that, together with the S^5 part of the metric, can be considered as the boundary of a ten dimensional gravity theory in Minkowski space-time. We will refer to the interior of AdS_5 as the bulk.
We thus have two different types of closed strings: closed strings propagating in flat ten-dimensional space-time, whose dynamics is described by free type IIB lowSUGRA, and closed strings propagating in the near horizon region. If we take the decoupling limit we get that both types of closed strings decouple from each other. This can be seen in this way: since our solution 3.22 is asymptotically flat, a process happening at distance r with energy $E(r)$, appears to an observer as having energy

$$E = \sqrt{-g_{00}}E(r) = \frac{1}{(1 + \frac{L^4}{r^4})^{\frac{1}{4}}}E(r), \tag{3.27}$$

the crucial observation is that E can be arbitrarily small for any value of $E(r)$ if r is very small. This means that the low energy limit, in this perspective, contains a sector besides the lowSUGRA one, where there are all the possible states, with any energy, produced in the near horizon region $r << L$. This means that the two types of closed strings decouple from each other.

To summarize, the background consists of two different regions: a near horizon region and an asymptotically flat region. The dynamics of the closed strings in asymptotically flat space-time are described by type IIB lowSUGRA in ten dimensional flat space-time, while the strings in the bulk region are described by fluctuations about the $\mathrm{AdS}_5 \times S^5$ solutions of IIB lowSUGRA. When we take the limit $\alpha' \to 0$, both types of closed strings decouple from each other.

Now it is time to merge this two perspectives; in both we get free type IIB lowSUGRA in ten dimensional Minkowski space-time so, since the two perspectives should be equivalent descriptions of the same physics, Maldacena [46] conjectured that full $D = 4$ $\mathcal{N} = 4$ SYM with gauge group $SU(N)$[10] and YM coupling $g^2_{YM} = 2\pi g_s$ is physically equivalent to type IIB lowSUGRA in $\mathrm{AdS}_5 \times S^5$ with radius L and N unit of $F_{(5)} = dC_{(4)}$ R-R flux[11]. Moreover, relaxing the low energy limit leads to the statement of AdS/CFT correspondence conjecture:

full $D = 4$ $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ and YM coupling $g^2_{YM} = 2\pi g_s$ is physically equivalent to full type IIB string theory on $\mathrm{AdS}_5 \times S^5$ with radius L satisfying the relation $2g^2_{YM}N = \frac{L^4}{\alpha'^2}$ with N unit of $F_{(5)} = dC_{(4)}$ R-R flux and string coupling g_s.

We will refer to the SYM theory as the CFT side on the correspondence while we will refer to the superstring theory as the AdS side.
A little bit of discussion is in order now. The interesting parameters of the two dual theories are, from the AdS side, the string coupling, g_s, and the ratio between the radius of the background space-time and the characteristic string length[12] $\frac{L}{l_s} = \frac{L}{\sqrt{2\alpha'}}$ and, from the CFT side, the YM coupling constant, g_{YM}, and N. These parameters are linked by the conjecture

$$g^2_{YM} = 2\pi g_s, \quad 2g^2_{YM}N = 2\lambda = \frac{L^4}{\alpha'^2}; \tag{3.28}$$

note that while the first of these equations involves g_{YM} the second one involves the 't Hooft coupling $\lambda = g^2_{YM}N$. If the AdS/CFT conjecture holds, all the physics of one description is mapped onto all the physics of the other one. This is very peculiar since in this way, we can map a possible candidate for a theory of quantum gravity, type IIB superstring theory, to a field theory without any gravitational degrees of freedom. Moreover, the AdS/CFT correspondence is a realisation of the holographic principle: the information of the five dimensional theory in AdS_5 obtained from compactification of type IIB string theory on S^5 is mapped to a four dimensional theory which lives on the conformal boundary of the five dimensional AdS_5 space-time.
The bold statement is called strongest AdS/CFT correspondence, although it says

[10] Here there is a little caveat: the low energy theory of a stack on N coincident D_3-branes is a gauge theory with gauge group $U(N)$ not $SU(N)$; however, turn out that $U(1)$ subgroup degrees of freedom decouples from the $SU(N)$ degrees of freedom and they cannot propagate into the bulk of AdS_5.

[11] The $F_{(5)}$ R-R flux enters in the calculation of the harmonic function $H_3(r)$ as explained at pag. 56 equation 2.64.

[12] Remember that $l_s = \sqrt{2\alpha'}$.

something very interesting and stimulates new ideas, is very difficult to use it to perform explicit calculations for generic values of the parameters. It is necessary to lessen the strength, but not the importance, of the proposed AdS/CFT correspondence by taking certain limits on both sides. We will see in a while that in this way we obtain more tractable forms of the correspondence and we can use them to study the strong coupling regime of one theory studying the computable weak coupling perturbative behaviour of the correspondent one.
Since currently string theory is best understood in the perturbative regime, it is useful to specialize the string theory side of the correspondence to weak coupling, $g_s << 1$, while keeping $\frac{L}{\sqrt{2\alpha'}}$ constant. At leading order in g_s, the AdS side reduces to classical string theory, in the sense that we take only tree level diagrams into account and not the entire genus expansion. Thanks to 3.28 we learn that the CFT side must have $g_{YM} << 1$ and λ remains constant; in other words we are taking the large N limit while keeping fixed the 't Hooft coupling constant and this corresponds to the planar limit of the gauge theory. This is called the strong form of AdS/CFT correspondence and is a concrete realisation of the idea of 't Hooft that the planar limit of a quantum field theory is a string theory as we saw in the first paragraph of this chapter.
Since we could be interested in strongly coupled field theories, we take the limit $\lambda \to \infty$ together with the large N limit: this implies that $\frac{\sqrt{2\alpha'}}{L} \to 0$. This means that the string length is small compared to the radius of the background space-time: this is the point particle limit of type IIB string theory, which is given by type IIB lowSUGRA as we know. This is the weak form of AdS/CFT correspondence. The three form of AdS/CFT correspondence are summarized in the following table.

AdS/CFT form	CFT side	AdS side
Strongest	any N and any λ	$g_s \neq 0$ and $\sqrt{2\alpha'}/L \neq 0$
Strong	large N and fixed λ	$g_s = 0$ and $\sqrt{2\alpha'}/L \neq 0$
Weak	large N and large λ	$g_s = 0$ and $\sqrt{2\alpha'}/L = 0$

Table 3.1. *The three forms of AdS/CFT correspondence. In the strongest form we have the full SYM theory and the full type IIB superstring theory; this form is difficult to use since on the AdS side we have the full genus expansion while in the CFT side we have the full SYM dynamics. In the strong form we have the large N limit of the SYM theory in the CFT side and the classical limit of the type IIB superstring theory in the AdS side; this is a more tractable form since the string theory does not contain the full genus expansion and on the CFT side we have to consider only the planar diagrams. In the weak form we have a strong coupled SYM theory in the CFT side and a classical supergavity theory in the AdS side; this form is very useful to calculate observables of the SYM theory since the AdS side contains a classical supergravity theory.*

In the end we have learned that AdS/CFT correspondence has to be considered as strong-weak duality: when the CFT side is strongly coupled the dual string theory reduces to its classical low effective supegravity behavior.

Let us give a first obvious check of the strongest form conjecture: do symmetries match in the right way in the two sides of the correspondence? The answer is yes, symmetries match but let us see how. From the CFT side we have an $\mathcal{N} = 4$

SYM theory; this theory is also conformal, its symmetry group is $SO(4,2)$, and so has thirtytwo supercharges. Moreover we know that a SUSY theory has also an R-symmetry group and in the case of $\mathcal{N}=4$ this is $SU_R(4) \simeq SO(6)$. From the AdS side we have type IIB superstring theory: this has thirtytwo supercharges and since the background space-time is $\mathrm{AdS}_5 \times S^5$ the symmetries $SO(4,2)$ and $SO(6)$ are realized as the isometries of the background space-time.

3.3.2 Holographic map

In the previous paragraph we had established the right statement of AdS/CFT correspondence. Now we need a map between the observables of the two theory and so a prescription for comparing physical quantity. The term "holographic" in the title of this paragraph is because, since AdS/CFT correspondence is a realization of holographic principle, is called simply holography.
We refer to the fields in five dimension AdS as bulk fields and to the CFT fields a boundary fields. We assume that the interaction of the bulk fields is described by an effective action, S_{AdS}, that in most applications is a supergravity one. Moreover, we call L_{CFT} the lagrangian of the CFT theory side. The crucial point of the correspondence is that a field in AdS is associated with an operator in the CFT side with the same quantum numbers; more precisely, from the four dimensional theory point of view every operator O is associated to a source ϕ_0 that is the boundary value of a bulk field. So in L_{CFT} will appear terms of the form $\int d^4xO\phi_0$ and taking functional derivatives with respect to ϕ_0 of the CFT partition function we get Green's functions for the operator O. Let us understand better how this works. For simplicity let consider a scalar bulk field $\phi(z,x)$ with mass m^2: its action in AdS_5 space-time is

$$S \simeq \int d^4x dz \sqrt{-g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2], \tag{3.29}$$

where $g^{\mu\nu}$ is the inverse metric of AdS_5 in Poincaré coordinates. The Klein-Gordon equation associated is

$$\frac{1}{L^2}\Big(z^2\partial_z^2 - 3z\partial_z + z^2\eta_{\mu\nu}dx^\mu dx^\nu\Big)\phi - m^2\phi = 0, \tag{3.30}$$

to solve this equation is convenient to perform a Fourier decomposition in the x^μ directions and to consider a plane wave of the form $\phi(z,x) = e^{ip^\mu x_\mu}\phi_p(z)$; inserting this ansatz in 3.30 we get the equation that determines the z-dependent part

$$z^2\partial_z^2\phi_p(z) - 3z\partial_z\phi_p(z) - (m^2L^2 + p^2z^2)\phi_p(z) = 0. \tag{3.31}$$

This equation admits power-like solutions $\phi_p(z) \simeq z^\Delta$, providing that Δ satisfies the quadratic equation

$$L^2m^2 = \Delta(\Delta - 4), \tag{3.32}$$

let us call Δ_+ and Δ_- its two roots and by definition $\Delta_+ > \Delta_-$ and $\Delta_- = 4 - \Delta_+$; near the boundary, $z \to 0$, we can expand the full solution $\phi(x,z)$ as

$$\phi(x,z) \simeq \phi_0 z^{\Delta_-} + \phi_v z^{\Delta_+} + sub.\ term\ in\ z. \tag{3.33}$$

turns out that by dimensional analysis, we may identify the ϕ_v as vacuum expectation value for a dual scalar field theory operator O with scaling dimension Δ_+, and ϕ_0 as source for this operator. Equation 3.32 provides a relation between the scaling dimension of the field theory operator O and the mass of the dual supergravity field $\phi(z, x)$.
These arguments can be generalized and applied to fields with higher spin. In the following table are reported the scaling dimension-mass relations.

Type of field	**Relation between m and Δ**
scalars, massive spin two fields	$m^2L^2 = \Delta(\Delta - 4)$
massless spin two fields	$m^2L^2 = 0, \Delta = 4$
p-form fields	$m^2L^2 = (\Delta - p)(\Delta + p - 4)$
spin $\frac{1}{2}$, spin $\frac{3}{2}$	$\lvert m\rvert L = \Delta - 2$
rank s symmetric traceless tensor	$m^2L^2 = (\Delta + s - 2)(\Delta - s - 2)$

Table 3.2. *Table showing the relation between the mass of the bulk field and Δ. Note that not necessary Δ is the scaling dimension of the dual CFT side operator, however, this is certainly intimately linked to it as we have seen in the case of a scalar bulk field.*

In the picture of this paragraph, AdS/CFT correspondence can be formulated using generating functionals language:

$$Z_{QG}[\phi, \phi'...]\Big|_{\phi_0, \phi'_0, ...} = Z_{CFT}[O\phi_0, O'\phi_0, ...]; \tag{3.34}$$

let us explain what does it mean 3.34. On the left side we have a QG generating functional with fields $\phi, \phi', ...$ and with boundary conditions $\phi_0, \phi'_0, ...$; on the other hand we have a CFT field theory generating functional with operators $O, O'...$ dual to the fields $\phi, \phi', ...$ and $\phi_0, \phi'_0, ...$ play the role of external sources. 3.34 is the starting point for the holographic calculation of correlation functions of CFT operators O. Introducing for all composite operators O_i on the field theory side the corresponding sources $\phi_{0,i}$, we obtain correlation functions from the generating functional Z_{CFT} by taking functional derivatives with respect to the sources $\phi_{0,i}$. The crucial point is that 3.34 provides an alternative way to calculate correlation functions using Z_{QG} instead of Z_{CFT}. Obviously this is very complicated since we do not know Z_{QG}; however in the case of weak form of AdS/CFT correspondence we know that Z_{QG} reduces to a supergravity generating functional Z_{SUGRA}. In this case there exist, in principle, a well established procedure:

1. Determine the bulk field ϕ which is dual to the operator O of scaling dimension Δ and compute Z_{SUGRA} by reducing type IIB supergravity;
2. Solve the supergravity equations of motion for ϕ, subject to the boundary condition ϕ_0 for $z \to 0$;
3. Insert the solution $\bar{\phi}$ into the supergravity action;
4. Take variational derivatives with respect to ϕ_0 to obtain correlation functions.

In the end, in the case of weak form AdS/CFT correspondence the calculation of CFT side correlation functions amounts to computing tree level diagrams on the gravity side. These tree level diagrams in AdS space are called Witten diagrams and some examples are reported below.

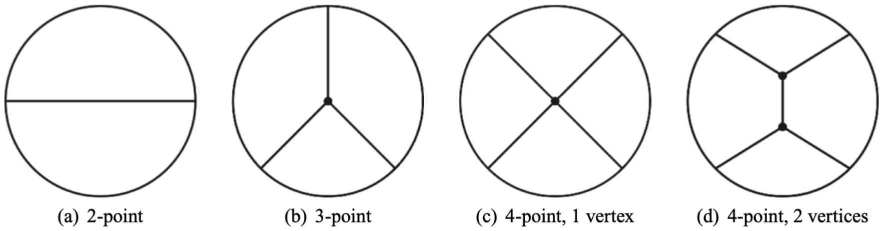

Figure 3.5. *Examples of Witten diagrams. The circle represent the boundary of AdS space-time while its interior is the bulk. Diagram (a) contains only one boundary-to-boundary propagator, diagrams (b) and (c) contain boundary-to-bulk propagators and diagram (d) contains also one bulk-to-bulk propagator. Figure taken from [42]*

The external sources $\phi_{0,i}$ of composite operators O_i are located at the conformal boundary of AdS space, which is represented by the circle in Figure 3.5; the bulk of AdS spacetime is given by the interior of the circle. Propagators depart from the external sources either to another boundary point (boundary-to-boundary propagators) or to an interior interaction point (boundary-to-bulk propagators); the structure of this interior interaction points is ruled by the interaction terms in the supergravity action and two interior interaction points may be connected (bulk-to-bulk propagators).

We will not go further in this interesting arguments but we refer to [42] par. 5.4 for more details.

3.4 Less superconformal gauge theories and Sasaki - Einstein manifolds

In the previous sections we have learned that AdS/CFT correspondence links a IIB superstring theory to a $D = 4$, $\mathcal{N} = 4$ conformal SYM theory. However, this theory contains too many supercharges to be of phenomenological interest: it contains only one supermultiplet and hence all the fields have to be in the same representation of the gauge group; obviously this is not what happens since gauge bosons and matter fields are in different representations of the gauge groups of SM. So, we now face the problem to extend AdS/CFT correspondence to potentially phenomenologically more interesting gauge theories; we do not want to break all the supercharges because we are not able to build up a string theory without SUSY but we also do not want a theory that contains all the supercharges contained in a $\mathcal{N} = 4$ SUSY theory. To extent the correspondence, the idea is to consider the superstring theory in a different background space-time however, since we want to preserve the conformal symmetry, our deformations cannot be done on AdS space-time but we have to work on S^5.

3.4.1 Sasaki-Einstein and Calabi-Yau manifolds

We know that the important point of AdS/CFT correspondence is to consider D_3-branes into a flat space-time and taking the limit near the D3-branes we obtain the background $\mathrm{AdS}_5 \times S^5$ then relaxing the low energy limit the correspondence between a string theory in $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory emerges. Now, the crucial observation is that the flat space-time where the D3-branes are embedded[13] can be written as $\mathbb{M}^{3,1} \times \mathbb{R}^6$ where we take the metric of $\mathbb{R}^6$ of the form

$$ds^2_{\mathbb{R}^6} = dr^2 + r^2 ds^2_{S^5}, \tag{3.35}$$

this is a cone metric and so $\mathbb{R}^6$ can be thought as the cone with base S^5. Hence, to formulate some extentions of the original AdS/CFT correspondence we can take, as background where embedding our D3-branes, the space-time $\mathbb{M}^{3,1} \times C(X_5)$, where $C(X_5)$ is the cone over the space-time X_5 which metric is

$$ds^2_{C(X_5)} = dr^2 + r^2 ds^2_{X_5}, \tag{3.36}$$

and so the original correspondence can be generalized in favor of the statement that a type IIB superstring theory in $\mathbb{M}^{3,1} \times X_5$ is physically equivalent to a suitable conformal SUSY field theory.
Important constrains for $C(X_5)$ arise from the requirement that the field theory is supersymmetric; the necessary structure for $C(X_5)$, in order that the field theory preserves some supercharges, is a Calabi-Yau structure: $C(X_5)$ has to be a Calabi-Yau manifold with complex dimension three. The interesting point is that in mathematics there exists a special class of manifolds whose cone has Calabi-Yau structure: Sasaki-Einstein manifolds. When $C(X_5)$ is Calabi-Yau cone we find a duality between string theory in $\mathrm{AdS}_5 \times X_5$ and the conformal field theory living on N D3-branes sitting in the conic singularity of $C(X5)$. These are in general $\mathcal{N} = 1$ supersymmetric gauge theories.

In the following of this paragraph we will focus on a brief review of Calabi-Yau and Sasaki-Einstein manifolds; see Appendix C for a short but useful survey of complex geometry based on [59].

Calabi-Yau manifold

In full generality, a Calabi-Yau (CY) manifold of dimension m is a complex manifold with Kähler structure and with trivial canonical bundle. For a compact Kähler manifold M with dimension m exist at least five equivalent prescriptions that make it a Calabi-Yau manifold:

1. has vanishing Ricci form;

2. has vanishing first Chern class;

3. has holonomy group contained into $SU(m)$;

4. has trivial canonical bundle;

[13] We have to pay attention: the Minkowski $\mathbb{M}^{3,1}$ is in fact the worldwolume of the D3-branes.

5. admits a globally defined and nowhere vanishing holomorphic m-form.

In 1954 Calabi formulated his conjecture [64] and only in 1976 Yau was able to give a proof [65]; Calabi-Yau theorem says that a compact Kähler manifold M with trivial first Chern class admits a unique Ricci flat Kähler metric. However, for our purpose we have to consider non compact Calabi-Yau manifolds: in the original AdS/CFT correspondence we consider $\mathbb{R}^6$ as the cone over S^5 but $\mathbb{R}^6 \simeq \mathbb{C}^3$ and $\mathbb{C}^3$ is a non compact Calabi-Yau threefold so it is reasonable to think that, in general, a Calabi-Yau threefold cone over some Sasaki-Einstein five manifold is non compact. Let us characterize a little Calabi-Yau manifolds, in particular we are interested in its cohomology: thanks to its Kähler structure and to its trivial canonical bundle we have only two independent Hodge number $h^{1,1}$ and $h^{1,2}$. This characterization led to the so-called mirror symmetry [66]: let us consider a CY manifold, if we exchange $h^{1,1}$ and $h^{1,2}$ we obtain a new CY manifold[14].

Sasaki-Einstein manifold

It is now time to study a little the base of the CY cone threefold: Sasaki-Einstein (SE) five dimensional manifold. Let us explain what it means Sasaki and Einstein. Sasakian geometry is the odd dimensional analogue of Kähler geometry in the sense that the latter has symplectic structure while the former has contact structure. Einstein geometry are manifolds such that their metric satisfies $R_{\mu\nu} = \lambda g_{\mu\nu}$. These two requires make the cone a CY manifold; hence, the most straightforward definition of SE is the following: a manifold is SE if and only if its cone is CY. The canonical example of a Sasaki-Einstein manifold is the odd dimensional sphere S^{2m-1}, equipped with its standard Einstein metric; its cone is $\mathbb{C}^m$ that is a CY manifold equipped with the flat metric. In general the cone of a Sasaki-Einstein manifold X has the metric

$$ds^2 = dr^2 + r^2 ds_X^2 \tag{3.37}$$

where $r \in \mathbb{R}^+$ and $r\partial_r$ is the generator of the homothetic action on the cone called Euler vector. A Sasaki-Einstein manifold inherits a number of geometric structures from the Kähler structure of its cone; in particular, an important role is played by the Reeb vector field. This may be defined as $\xi := J(r\partial_r)$, where J denotes the complex structure of the cone. The Reeb vector field is a Killing vector field and there exists a classification of Sasaki-Einstein manifolds according to the global properties of the orbits of the Reeb vector field:

- regular: the orbits are closed and the group action is globally free[15]. In this case the action gives a $U(1)$ symmetry and the length of the orbits are all the same. We have a principal bundle[16] over a four dimensional CY manifold;
- quasi regular: the orbits are closed, but the group action is not globally free. Also in this case there is a $U(1)$ symmetry but now the length of the orbits

[14] More rigorously, there exist isomorphisms between the cohomology group $H^{1,1}$ and $H^{1,2}$ and we can think that the complex structure is exchanged with the Kähler one

[15] Free action means that if, given g and h in a group G, the existence of an x in a manifold X with $gx = hx$ implies $g = h$.

[16] A principal bundle is a fiber bundle together with a continuous right action of a group on the total space that preserves its fibers.

are not all the same: there is at least one point where the length is reduced and we have a principal orbibundle[17] over a four dimensional CY manifold;

- irregular: the orbits are not closed.

The five dimensional regular SE manifolds are completely classified by Friedrich and Kath [71]. Until 2004 no explicit examples of non trivial strictly quasi regular Sasaki-Einstein manifolds were known, and it was not known whether or not irregular Sasaki-Einstein manifolds even existed; infact Cheeger and Tian conjectured [68] that they did not exist. However in 2004 Gauntlett, Martelli, Sparks and Waldram [69] found that there exist a countably infinite number of Sasaki-Einstein metrics $Y^{p,q}$ on $S^2 \times S^3$, labelled naturally by $p, q \in \mathbb{N}$ where $gcd(p, q) = 1$, so they are coprime, and $q < p$. Turns out that $Y^{p,q}$ is quasi regular if and only if $4p^2 - 3q^2$ is the square of a natural number, otherwise it is irregular. These metrics are constructed explicitly[18] and they have cohomogeneity one, meaning that the generic orbit under the action of the isometry group has real codimension one. Thanks to Smale's classification of five dimensional [70] manifolds follows that $Y^{p,q}$ is diffeomorphic to $S^2 \times S^3$; the volume of this manifold is given by

$$Vol(Y^{p,q}) = \frac{q^2(2p + \sqrt{4p^2 - 3q^2})}{3p^2(3q^2 - 2p^2 + \sqrt{4p^2 - 3q^2})}\pi^3. \tag{3.38}$$

There exist also explicit cohomogeneity two Sasaki-Einstein five dimensional manifolds [72], [73]; these are a countably infinite number of Sasaki-Einstein metrics $L^{(a,b,c)}$ on $S^2 \times S^3$, labelled naturally by $a, b, c \in \mathbb{N}$ where $a \leq b$, $c \leq b$, $gcd(a, b, c, a+b-c) = 1$, $gcd(\{a, b\}, \{c, d\}) = 1$. Moreover, turns out that $L^{(p-q,p+q,p)} = Y^{p,q}$. These are in general irregular.

It is important to stress out that for general CY cone we are not able to know the worldvolume gauge theory but we are able to know it, at least in some cases, if the CY threefold cone is toric. We will enter in some detail into toric geometry in the next chapter of this work; now let us consider a first explicit example of AdS/CFT with D3-branes probing a CY cone singularity due to Klebanov and Witten [75].

3.4.2 Klebanov-Witten model

Klebanov-Witten article [75] was a guiding work that, in 1998, extended the original AdS/CFT correspondence to $\mathcal{N} = 1$ SUSY gauge theory: the so-called Klebanov-Witten model. At that time they not know the infinite class of SE manifold $Y^{p,q}$ and $L^{(a,b,c)}$ but they found a second example of five dimensional SE manifold after S^5: the homogeneous space[19] $T^{1,1} = \frac{SU(2)\times SU(2)}{U(1)} \simeq \frac{SO(4)}{U(1)}$.

[17] It is an orbifold principal bundle. An orbifold is much like a smooth manifold but possibly with singularities of the form of fixed points of finite group actions. As a smooth manifold is a space locally modeled on euclidean space, an orbifold is a space that is locally modeled on smooth quotient space, quotient by a finite group; often this finite groups are cyclic ones.

[18] See [58] pag. 10.

[19] Homogeneous spaces are a pair (X, G) where X is a topological space and G is a group that acts transitively on X; this means that X is non empty and that for each pair $x, y \in X$ there exists a g in G such that $gx = y$.

The main idea of Klebanov-Witten model is to construct the theory in a way that the mesonic moduli space is the conifold. Let us explain what conifold is: consider the three dimensional complex variety with singular origin

$$\mathcal{C} = \left\{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \middle| z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \right\}, \tag{3.39}$$

this is a cone since it admits an homothetic action of the form $z_i \to t z_i$ with $i = 1, 2, 3, 4$ and $t \in \mathbb{R}^+$. In [76] it is shown that the conifold $\mathcal{C}$ admits a CY metric so a metric that is Kähler Ricci flat. This implies that the base of the conifold is a five dimensional SE manifold X_5. To see this, let us consider the conifold expressed by the relation

$$z_1 z_2 - z_3 z_4 = 0, \tag{3.40}$$

it is obvious that there exists a $U(1)$ action given by $z_i \to e^{i\theta} z_i$ with $i = 1, 2, 3, 4$; this action is inherited by the cone's base and quotient it out makes possible to show that X_5 is a $U(1)$ fibration over $S^2 \times S^2$. We can parametrize the two spheres with angles (α_a, β_a) with $a = 1, 2$ and the $U(1)$ fiber with and angle $\phi \in [0, 4\pi)$, then turns out that the five dimensional angular part of the conifold metric written in the variables $(\alpha_a, \beta_a, \phi)$ gives a SE structure: X_5.
However, in their work Klebanov and Witten started from the quotient description of X_5. Starting from 3.40 we can write

$$\begin{aligned} z_1 &= A_1 B_1; \\ z_2 &= A_2 B_2; \\ z_3 &= A_1 B_2; \\ z_4 &= A_2 B_1, \end{aligned} \tag{3.41}$$

to parametrize the solutions; note that nothing change if we apply the transoformations

$$A_i \to \lambda A_i, \quad B_j \to \lambda^{-1} B_j, \tag{3.42}$$

with $i, j = 1, 2$ and $\lambda \in \mathbb{C}^*$: this is an $SU(2) \times SU(2)$ symmetry of the conifold where one $SU(2)$ acts on A_i while the other $SU(2)$ acts on B_j. If we write $\lambda = s e^{i\theta}$ with $s \in \mathbb{R}^+$ and $\theta \in \mathbb{R}$ and if we consider $z_i \neq 0$, s can be selected to set

$$|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2, \tag{3.43}$$

and the conifold is obtained by quotient by the remaining $U(1)$ action given by

$$A_i \to e^{i\theta} A_i, \quad B_j \to e^{-i\theta} B_j. \tag{3.44}$$

From this point of view the angular manifold part, X_5, can be obtained by quotient by the scaling action $z_i \to s z_i$ by setting 3.43 to one: this implies $s = 1$ and so the scaling action is eliminated. So considering the $SU(2) \times SU(2)$ action without scaling action we should have the manifold $S^3 \times S^3$; then quotient by the $U(1)$ action 3.44 we obtain a five dimensional manifold given by

$$X_5 = \frac{SU(2) \times SU(2)}{U(1)} := T^{1,1}. \tag{3.45}$$

Klebanov and Witten continue with the identification of the gauge theory. Let us consider a stack on N D_3-branes at the conical singularity of $\mathbb{M}^{3,1} \times \mathcal{C}$; we want that the moduli space of the gauge theory living on the worldvolume of the D_3-branes to be the conifold. For this purpose is natural to introduce four chiral superfields A_1, A_2, B_1, B_2 which are reminiscent of the geometric description given above; this implies that the gauge theory must have the global symmetry $SU(2) \times SU(2) \times U(1)_R$ as happens for its geometric counterpart. Since in four dimension only $\mathcal{N} = 1$ SUSY gauge theory are chiral, the dual field theory to superstring IIB theory on $AdS_5 \times T^{1,1}$ must be a $\mathcal{N} = 1$ SUSY gauge theory. We know that in $\mathcal{N} = 1$ theories there is always a global $U(1)$ symmetry: the R-symmetry. Correspondingly, the only isometry that SE manifolds have in a systematically way is the one induced by the Reeb vector field. Hence, is quite naturally to identify these two symmetries as dual under AdS/CFT correspondence: this is a general feature. Moreover, turns out that the gauge group must be $U(N) \times U(N)$ and the four chiral superfields transform as bifundamental under these gauge groups: they are $N \times N$ matrices; furthermore the anomaly cancellation condition for $U(1)_R$ and the symmetry of the theory implies that the chiral field has R-charge $R = \frac{1}{2}$. These values force the superpotential of the theory to be quartic since we must have R-charge 2 for the superpotential, $R[W] = 2$. This implies, for example, that the field theory is non renormalizable. Given the high symmetry of the conifold Klebanov and Witten were able to write down the superpotential by brute force without knowing the technology that we will introduce in the next chapter; there is only one gauge invariant operator singlet under the global symmetry $SU(2) \times SU(2)$ and with $R = 2$ is

$$W \propto \epsilon^{ij}\epsilon^{lk} Tr(A_i B_k A_j B_l) \tag{3.46}$$

where the trace is over gauge indexes.
Theories like this that have $\mathcal{N} = 1$ and fields transforming non trivially for exactly two gauge groups, precisely in the fundamental representation of one of these and in the antifundamental of the other, are called quiver gauge theories and their matter content, gauge groups and superpotential can be represented diagrammatically using the so-called quiver diagrams or simply quiver, specifying completely the theory. The quiver is build up in the following way:

- to each gauge group G_k we associate a node in the diagram;
- to each field in the fundamental of G_i and in the antifundamental of G_j we associate an arrow pointing from G_i to G_j.

For the Klebanov-Witten model is easy to see that the quiver is

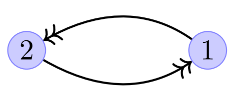

Figure 3.6. *Quiver diagram for Klebanov-Witten model: we have two gauge groups and four chiral bifundamental fields.*

Klebanov-Witten model furnishes a generalization of AdS/CFT correspondence to a less superconformal theory: type IIB superstring theory on $AdS_5 \times T^{1,1}$ is dual to $\mathcal{N} = 1$ SUSY quiver gauge theory with moduli space given by conifold[20]. However while $\mathcal{N} = 4$ SUSY theory is always conformal, $\mathcal{N} = 1$ SUSY theory is not; so the correspondence holds only at the fixed points of the SUSY gauge theory's beta function.
By explicit inspection, it is possible to calculate the volume of the $T^{1,1}$ manifold,

$$Vol(T^{1,1}) = \frac{16\pi^3}{27}, \tag{3.47}$$

this volume is linked by Gubser formula [77] to the central charge a that appears in trace anomaly B.6:

$$Vol(T^{1,1}) = \frac{\pi^3 N^2}{4a}. \tag{3.48}$$

The value of a can be computed using[21] [78]

$$a = \frac{3}{32}[3Tr(R^3) - Tr(R)], \tag{3.49}$$

where trace means that we have to sum over all the Weyl fermions of the theory and the R are the values of their R-charges[22]. For the Klebanov-Witten model we have four $N \times N$ chiral superfields and $2N^2$ gaugini so

$$a = \frac{3}{32}\left[3\left[4N^2\left(-\frac{1}{2}\right)^3 + 2N^2\right] - \left(4N^2\left(-\frac{1}{2}\right) + 2N^2\right)\right] = \frac{27N^2}{64}, \tag{3.50}$$

and it matches with Gubser formula. This is a general feature of this extension of AdS/CFT correspondence: the volume of the SE manifold is dual to the central charge a computed knowing the R-charges and using 3.49; they are related with Gubser formula. Viceversa, as found in 2003 by Intriligator and Wecht [79], if we do not know the R-charges we can find them maximizing the value of a considering the trial function

$$a_T(\vec{r}) = \frac{3}{32}[3Tr(R^3(\vec{r})) - Tr(R(\vec{r}))] \tag{3.51}$$

where $\vec{r} = (r_1, ..., r_m)$ and m is the number of independent $U(1)$ symmetry of the theory[23]. This procedure is called a-maximization and when the theory is conformal the linear trace term must vanishes. Thanks to Gubser formula this is equivalent to minimize the volume of the SE manifold, however, in general we do not know the metric of the Sasaki-Einstein manifold and so we can not calculate its volume; the right way is a-maximization, at least in the IR regime.

[20]This will be a general fact: given a type IIB superstring theory on $AdS_5 \times X_5$ the moduli space of the dual field theory will be the CY cone on X_5, $C(X_5)$.

[21]The value of c is computed in a similar way: $c = \frac{1}{32}[9Tr(R^3) - 5Tr(R)]$.

[22]Recall that for a vector multiplet the R-charge is conventionally 1, the charge of the gaugino, while for a a chiral multiplet it is the charge of the scalar so that the fermion chsrge is $r = R - 1$.

[23]This are in general the number of gauge group minus one: this is because each $U(N) = SU(N) \times U(1)$ and one of this $U(1)$ is not independent in the IR, region where we are interested.

Chapter 4

Brane tilings and quiver theories

At the end of the last chapter we have understood how to extend the original Maldacena's AdS/CFT correspondence to potentially more phenomenologically interesting chiral $\mathcal{N} = 1$, $D = 4$ SUSY gauge theories. We have seen that to do so we have to embed our stack of N D3-branes into a background space-time of the form $\mathbb{M}^{3,1} \times C(X_5)$ with $C(X_5)$ a Calabi-Yau cone over X_5 or, equivalently, to consider a type IIB superstring theory in $\mathrm{AdS}_5 \times X_5$ with X_5 a Sasaki-Einstein manifold. However, for general Calabi-Yau cone, we are not able to build up the worldvolume gauge theory living on the D_3-branes configuration and so we need one more constrain: we restrict the class of Calabi-Yau varieties that we consider to those that are toric. The toric condition simplifies a lot the geometric description of the Calabi-Yau cone giving the possibility to compute non trivial results in string theory; nevertheless we have not the complete control of the geometry since there exist a few known examples of explicit metrics.
It is not simple to explain in a few words what toric geometry is because it has many equivalent and complementary descriptions: for example it may be approached in algebraic geometry using cones, fan and homogeneous coordinates; in symplectic geometry thanks to moment maps and Kähler quotients or in the context of the so-called Gauged Linear Sigma Model (GLSM) as the SUSY moduli space of the theory or again as a Delzant-like construction over a polytope. Anyway, a general feature of an n-dimensional toric variety $\mathcal{M}$ is the presence of an algebraic torus action in the sense that the algebraic torus $\mathbb{T}^n = (\mathbb{C}^*)^n$ is a dense open subset and there is an action $\mathbb{T}^n \times \mathcal{M} \to \mathcal{M}$ that extends the action of the algebraic torus on itself. The greatest point in favor of toric geometry is that geometry of a toric variety is fully determined by combinatorics
For the case of a toric Calabi-Yau variety the information about the geometry is summarized in the so-called toric diagram which is a polytope embedded in a $\mathbb{Z}^2$-lattice. Moreover, when we consider a toric Calabi-Yau threefold cone the action of the algebraic torus enlarges the isometry group of the variety from $U(1)$, the action induced by the Reeb vector field, to $U(1)^3$. So, it is possible to T-dualize along the cycles that identify the algebraic torus and in the case of Calabi-Yau threefolds this means applying T-duality in two directions and so D3-branes are

mapped in D5-branes. However we must remember that T-duality interchanges also geometry and the KR field; this means that the Calabi-Yau background is mapped to the so-called NS5-branes that are the Hodge dual of fundamental string and so it is charged under the KR field. This duality between a system of D3-branes sitting at the singularity of a Calabi-Yau toric threefold cone and a system of D5-branes and NS5-branes is the starting point for the construction of the so-called brane tiling which we will see in great detail in the following. These brane tilings are related to the dual graph of the toric diagram and they furnish a incredible method to build up the gauge theory dual to the D3-branes configuration and are a powerful construction since studying what we can do on this diagram we can better understand the physics of this duality. Brane tilings contain all the information that we need to specify a $\mathcal{N} = 1$ SUSY gauge theory: matter content, gauge groups and the superpotential; these information can be redrawn in the so-called quiver diagrams: a network of node and oriented arrows.
In the following we will give first some notions and basic tools of toric geometry [80],[81],[82],[83],[84],[86],[93] and we will study the reflexive polytopes, a special class of toric diagrams. Then we will move on brane tilings [85],[88],[89],[90],[91],[92] studying their construction starting from the toric diagrams and viceversa, their physical interpretation and some tools about them; moreover we will undersand how to construct the quiver diagram starting from brane tiling and we will list the thirty quiver theories associated to reflexive toric diagrams.

4.1 Toric geometry

We have mentioned that toric varieties have many faces that are complementary; we will start studying the homogeneous coordinates approach which hides a little the connection with physics but which has the advantage of giving a good mathematical vision, after which we will see the approach of momentum maps and polytopes which will make us understand the connection with the GLSM.

4.1.1 Homogenous coordinates approach to toric varieties

We begin with the simplest construction of toric variety: as generalization of weighed projective space. This approach to toric geometry is due to Cox. Recall first the definition of the $m-1$-dimensional weighed projective space

$$\mathbb{CP}^{m-1} = \frac{\mathbb{C}^m \setminus \{0\}}{\mathbb{C}^*} \tag{4.1}$$

where the quotient by $\mathbb{C}^*$ is implemented thanks to the identification $(z_1, ..., z_{m-1}) \sim (\lambda^{i_1} z_1, ..., \lambda^{i_{m-1}} z_{m-1})$ where $\lambda \in \mathbb{C}^*$ and $(i_1, ..., i_{m-1})$ are the weights of the coordinates. A n-dimensional toric variety $\mathcal{M}$ is the generalization where we quotient by more than one $\mathbb{C}^*$ action and the set that we subtract is a subset U_Σ which contains not only the origin

$$\mathcal{M} = \frac{\mathbb{C}^m \setminus \{U_\Sigma\}}{(\mathbb{C}^*)^{m-n} \times \Gamma}, \tag{4.2}$$

where Γ is an abelian group; this variety has an algebraic torus action given by $(\mathbb{C}^*)^{m-m+n} = (\mathbb{C}^*)^n$. Let us see how toric variety emerges using the Cox's approach.

Let M and N be a dual n-dimensional lattices isomorphic to $\mathbb{Z}^n$ and consider the vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ to be the subspace of $\mathbb{R}^n$ spanned, respectively, by vectors in M and N. We define the Strongly Convex Rational Polyhedral Cone (SCRPC) $\sigma \in N_{\mathbb{R}} \subset \mathbb{R}^n$ as the set

$$\sigma := \left\{ \sum_{i=1}^{m} a_i \vec{v}_i \middle| a_i \in \mathbb{R}, a_i \geq 0, \vec{v}_i \in \mathbb{Z}^n \ \forall i \right\} \tag{4.3}$$

for a finite number of vectors v_i and satisfying the condition $(\sigma) \cap (-\sigma) = \{0\}$. Let us analyze this definition: consider an n-dimensional lattice $N \simeq \mathbb{Z}^n$, a SCRPC is an n or lower dimensional cone in $N_{\mathbb{R}}$ with the origin of the lattice as its apex, bounded by hyperplanes (polyhedral) with its edges spanned by lattice vectors (rational) and such that it does not contain complete lines (strongly convex). The dimension of a SCRPC σ is the dimension of the smallest subspace of $\mathbb{R}^n$ containing σ; there are two important concepts that we need to introduce now:

- edges: these are the one dimensional faces of σ, the vectors $\vec{g}$ associated to the edges are the generators of σ;
- facets: these are the codimension one faces.

Examples are shown in figure below.

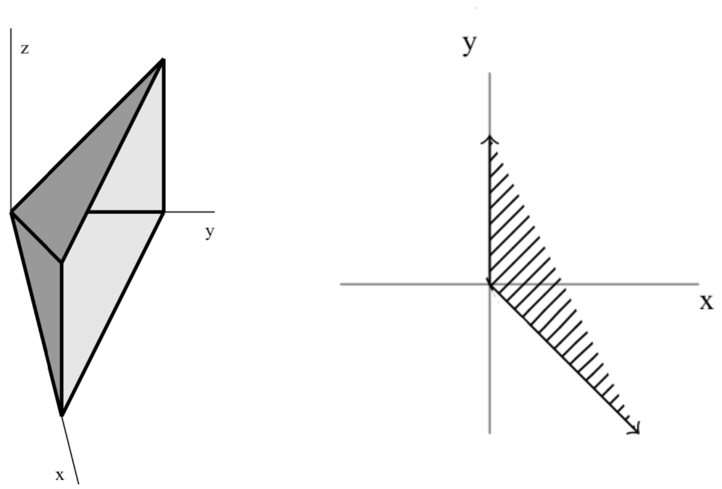

Figure 4.1. *Examples of SCRPCs. Left: SCRPC in $\mathbb{R}^3$, its one dimensional faces are identified by the vectors $\vec{g}_1 = (1,0,0), \vec{g}_2 = (0,1,0), \vec{g}_3 = (0,1,1), \vec{g}_4 = (1,0,1)$ in $\mathbb{Z}^3$. Right: SCRPC in $\mathbb{R}^2$, its one dimensional faces are identified by the vectors $\vec{g}_1 = (0,1), \vec{g}_2 = (1,-1)$ in $\mathbb{Z}^2$. Figure taken from [82].*

A collection Σ of SCRPCs in $N_{\mathbb{R}}$ is called fan if each face of a SCRPC in Σ is also a SCRPC in Σ and the intersection of two SCRPCs in Σ is a face of each. Examples of fan are reported in Figure 4.2.
At this point, let us consider a fan Σ and we call Σ_{1d} the set of one dimensional SCRPCs; let $\vec{v}_i$ with $i = 1, ..., m$ be the whole set of vectors generating the one

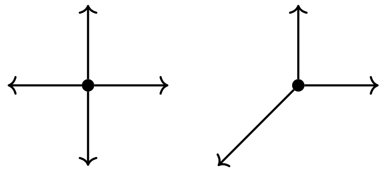

Figure 4.2. *Examples of fan. Left: fan of $\mathbb{CP}^1 \times \mathbb{CP}^1$; we have four one dimensional SCRPCs (the vectors) and four two dimensional SCRPCs (the quadrants). Right: fan of $\mathbb{CP}^2$; we have three one dimensional SCRPCs (the vectors) and three two dimensional ones (the trians).*

dimensional SCRPCs in Σ_{1d}[1]. To each vector $\vec{v}_i$ we associate an homogeneous coordinate $z_i \in \mathbb{C}$, from the resulting $\mathbb{C}^m$ we subtract the set

$$U_\Sigma := \bigcup_I \{(z_1, ..., z_m) | z_i = 0 \ \forall i \in I\}, \tag{4.4}$$

where the union is taken over all the sets having $I \subseteq \{1, ..., m\}$ for which z_i with $i \in I$ does not belong to a SCRPC in Σ. At this point we need to discuss how the $(\mathbb{C}^*)^{m-n} \times \Gamma$ acts on $\mathbb{C}^m$. First of all, let us clarify the nature of the abelian group Γ: this is given by

$$\Gamma := \frac{N}{\tilde{N}}, \tag{4.5}$$

where $\tilde{N} \subset N$ is the sublattice generated over $\mathbb{Z}^n$ by the vectors $\vec{v}_i$; in other words, vectors $\vec{v}_i$ not necessarily generate all N, in general they generate only $\tilde{N}$, a sublattice of N. If $\tilde{N} = N$, Γ is trivial and no takes part in the quotient. On the other hand, if Γ is no trivial our variety develops orbifold singularity. Hence, now we must know the action of the algebraic torus on $\mathbb{C}^m$. Let us consider the $n \times m$ matrix built up considering the m vectors $\vec{v}_i$ with n components[2]

$$V_i^k = \begin{pmatrix} v_1^1 & v_2^1 & \dots & v_m^1 \\ v_1^2 & \ddots & \dots & v_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ v_1^n & v_2^n & \dots & v_m^n \end{pmatrix}, \tag{4.6}$$

this induces a map $\phi : \mathbb{C}^m \to \mathbb{C}^n$ defined by

$$(z_1, ..., z_m) \mapsto \left(\prod_{i=1}^m z_i^{v_i^1}, ..., \prod_{i=1}^m z_i^{v_i^n} \right). \tag{4.7}$$

Thanks to the rank-nullity theorem[3] the dimension of the kernel of this map must have dimension $m - n$ and so we can identify it with $(\mathbb{C}^*)^{m-n}$, now it is simple

[1] Note that obviously m is equal to the number of one dimensional cone and so to the number of elements in Σ_{1d}

[2] Recall that $\vec{v}_i$ are m vectors belonging to $\mathbb{Z}^n$.

[3] Given T a linear map between two finite dimensional vector spaces V and W we have that $dim(Im(T)) + dim(ken(T)) = dim(V)$.

to see how $(\mathbb{C}^*)^{m-n}$ acts on $\mathbb{C}^m$: each $\mathbb{C}^*$ action is implemented like $(z_1, ..., z_m) \mapsto (\lambda^{Q_1^a} z_1, ..., \lambda^{Q_m^a} z_m)$ with $\lambda \in \mathbb{C}^*$ and so we have $m-n$ actions like this, where for each $a = 1, ..., m-n$ the charge vectors $Q^a = (Q_1^a, ..., Q_m^a)$ belong to the kernel of the map ϕ and so they must satisfy $m-n$ relations

$$\sum_{i=1}^{m} V_i^k Q_i^a = \vec{0}. \tag{4.8}$$

Roughly speaking, since $(\mathbb{C}^*)^{m-n} = Ker(\phi)$ it acts as usual, but with weights which are the components of the vectors belonging to the kernel of V_i^k. Quotient $(\mathbb{C}^*)^{m-n}$ out means taking the equivalence relations

$$(z_1, ..., z_m) \sim (\lambda^{Q_1^a} z_1, ..., \lambda^{Q_m^a} z_m) \tag{4.9}$$

for $a = 1, ..., m-n$.
To summarize, putting it all together, we can define the toric variety as

$$\mathcal{M} = \frac{\mathbb{C}^m \setminus \{U_\Sigma\}}{(\mathbb{C}^*)^{m-n} \times \Gamma}, \tag{4.10}$$

this is a n-dimensional variety, with a residual $(\mathbb{C}^*)^n \simeq U(1)^n$ action and the $(\mathbb{C}^*)^{m-n}$ action is quotient out by $m-n$ relations 4.9 with weight that satisfies relations 4.8. Let us give some examples to better understand the construction given above.

Exemple 1

Let us consider the right fan in Figure 4.2, this is a $\mathbb{Z}^2$ lattice. We have the three vectors $\vec{v}_1 = (0,1), \vec{v}_2 = (1,0), \vec{v}_3 = (-1,-1)$ that generate one dimensional cones so we have three homogeneous coordinates $(z_1, z_2, z_3) \in \mathbb{C}^3$; the set U_Σ is given by the origin. Since $m = 3$ and $n = 2$ we must have one $\mathbb{C}^*$ action and we can find the weight using 4.8:

$$\begin{aligned} &\vec{v}_1 Q_1 + \vec{v}_2 Q_2 + \vec{v}_3 Q_3 = (0,1)Q_1 + (1,0)Q_2 + (-1,-1)Q_3 = \\ &= (Q_2 - Q_3, Q_1 - Q_3) = \vec{0} \Rightarrow Q_1 = Q_2 = Q_3 = 1, \end{aligned} \tag{4.11}$$

and so the $\mathbb{C}^*$ action is implemented by the equivalence relation $(z_1, z_2, z_3) \sim \lambda(z_1, z_2, z_3)$. Finally we note that the vectors $\vec{v}_1$ and $\vec{v}_2$ generate the whole lattice $\mathbb{Z}^2$ and so Γ is trivial. In the end this fan corresponds to

$$\mathcal{M} = \frac{\mathbb{C}^3 \setminus \{0\}}{\mathbb{C}^*} \equiv \mathbb{CP}^2, \tag{4.12}$$

as we expected.

Example 2

Let us consider the left fan in Figure 4.2. We have the four vectors $\vec{v}_1 = (0,1), \vec{v}_2 = (0,-1), \vec{v}_3 = (1,0), \vec{v}_4 = (-1,0)$ that generate one dimensional cones so we have four homogeneous coordinates $(z_1, z_2, z_3, z_4) \in \mathbb{C}^4$; the set U_Σ is given by

$$(\{0,0\}, \mathbb{C}^2) \cup (\mathbb{C}^2, \{0,0\}). \tag{4.13}$$

The abelian group Γ is trivial since the vectors generate the whole lattice $N \simeq \mathbb{Z}^2$. Since $m = 4$ and $n = 2$ we expect to have a $(\mathbb{C}^*)^2$, charge vectors are given by

$$\begin{aligned}(0,1)Q_1^1 + (0,-1)Q_2^1 + (1,0)Q_3^1 + (-1,0)Q_4^1 = (Q_3^1 - Q_4^1, Q_1^1 - Q_2^1) = \vec{0};\\ (0,1)Q_1^2 + (0,-1)Q_2^2 + (1,0)Q_3^2 + (-1,0)Q_4^2 = (Q_3^2 - Q_4^2, Q_1^2 - Q_2^2) = \vec{0},\end{aligned} \tag{4.14}$$

so we obtain $Q_1^1 = Q_2^1 = 1, Q_3^1 = Q_4^1 = 0$ and $Q_3^2 = Q_4^2 = 1, Q_1^2 = Q_2^2 = 0$; we have two equivalence relations of the form $(z_1, z_2, z_3, z_4)_1 \sim (\lambda z_1, \lambda z_2, z_3, z_4)$ and $(z_1, z_2, z_3, z_4)_2 \sim (z_1, z_2, \lambda z_3, \lambda z_4)$. In the end we have two independent copies of $\mathbb{CP}^1$

$$\mathcal{M} = \frac{\mathbb{C}^4 \setminus ((\{0,0\}, \mathbb{C}^2) \cup (\mathbb{C}^2, \{0,0\}))}{(\mathbb{C}^*)^2} \equiv \mathbb{CP}^1 \times \mathbb{CP}^1, \tag{4.15}$$

again as we expected.

We now give some interesting properties about toric varieties and their fan:

- a fan Σ is smooth if every SCRPC in Σ is smooth, a SCRPC is smooth if is generated by a subset of a basis of $N \simeq \mathbb{Z}^n$;
- a fan Σ is simplicial if every SCRPC in Σ is simplicial, a SCRPC is simplicial if is generated by a subset of a basis of $\mathbb{R}^n$;

These conditions are important since if a fan is smooth the corresponding toric variety also is smooth and if a fan is simplicial the corresponding toric variety can have at most orbifold singularities[4]. We see immediately that the two spaces described by the fans in Figure 4.2 are smooth since every SCRPC is generated by a subset of a $\mathbb{Z}^2$ basis. An example of toric variety with orbifold singularities is the weighted projective space $\mathbb{CP}_{2,3,1}$; its fan is given by the Figure 4.3 below. Orbifold singularities can be removed by the so-called blow up procedure: roughly speaking, for a n-dimensional toric variety we replace the singular locus by $\mathbb{CP}^{n-1}$. It is now

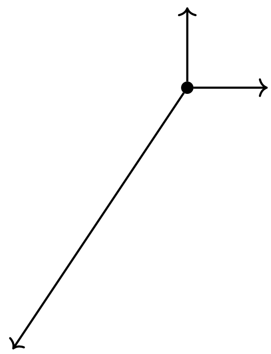

Figure 4.3. *Fan of* $\mathbb{CP}_{2,3,1}$ *we have* $\vec{v}_1 = (1,0), \vec{v}_2 = (0,1), \vec{v}_3 = (-2,-3)$. *It is not smooth but it is simplicial:* $\mathbb{CP}_{2,3,1}$ *has orbifold singularities.*

time to specialize to the case of our interest: CY threefolds. First of all in this case

[4]These results are shown, for example, in [83]

we have an $U(1)^3 \simeq (\mathbb{C}^*)^3$ action but there is more: the condition of trivial canonical bundle implies that all the vectors of the fan belong to the same hyperplane, so we can project on this hyperplane obtaining a two dimensional object whose convex hull take the name of toric diagram. The CY condition can be implemented also on the map ϕ, the triviality of the canonical bundle makes that the sum of the components of the charge vectors is zero

$$CY\ condition \Rightarrow \sum_{i=1}^{m} Q_i^a = 0\ \forall a \tag{4.16}$$

Example: the conifold $\mathcal{C}$

Consider the three dimensional fan given by the four vectors $\vec{v}_1 = (1,0,1), \vec{v}_2 = (0,0,1), \vec{v}_3 = (0,1,1), \vec{v}_4 = (1,1,1)$. We note that these vectors belong to the same hyperplane, so we can project out the third component: we obtain $\vec{w}_1 = (0,0), \vec{w}_2 = (1,0), \vec{w}_3 = (0,1), \vec{w}_4 = (1,1)$. Hence the conifold is a CY variety, its toric diagram is given below

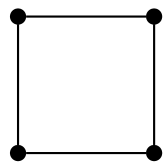

Figure 4.4. *Toric diagram of the conifold.*

Since we know that a CY toric variety must satisfy condition 4.16, let us check this. We have $m - n = 4 - 3 = 1$ charge vector given by relation 4.8

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} = \vec{0} \Rightarrow \begin{cases} Q_1 + Q_4 & = 0 \\ Q_3 + Q_4 & = 0 \\ \sum_{i=1}^4 Q_i & = 0 \end{cases} \tag{4.17}$$

a possible solution is $Q = (1, -1, 1 - 1)$ and the sum up to zero. As one can note the CY condition is automatically implemented by the fact that the vectors are coplanar. The action of the algebraic torus $\mathbb{C}^*$ is quotient out by the equivalence relation

$$(z_1, z_2, z_3, z_4) \sim (\lambda z_1, \lambda^{-1} z_2, \lambda z_3, \lambda^{-1} z_4); \tag{4.18}$$

let us consider the $\mathbb{C}^*$-invariant polynomials, by 4.18 we note that

$$x_1 = z_1 z_2,\ x_2 = z_1 z_4,\ x_3 = z_2 z_3,\ x_4 = z_3 z_4, \tag{4.19}$$

are invariant and this is the minimal basis with which to write all the $\mathbb{C}^*$ invariant polynomials. However, note that these polynomials are not independent but they must satisfy the relation

$$x_1 x_4 = x_2 x_3 \tag{4.20}$$

which is exactly the conifold relation 3.40.
This is a general algorithm to identify to which manifold a toric diagram belongs:

given the toric diagram we look at the equivalent relations that quotient out the algebraic torus action $(\mathbb{C}^*)^{m-n}$ and we construct a minimal basis of m $(\mathbb{C}^*)^{m-n}$-invariant polynomials; the relations that this polynomials must satisfy identifies the toric variety. Before giving further examples we now see the approach of moment map to toric geometry; this is because in the homogeneous coordinates approach the link with physics is quite mysterious in favor of a simpler mathematical treatment.

4.1.2 Moment maps approach and Delzant-like construction

Let us take a step back and consider a symplectic manifold $\mathcal{M}$ of real dimension $2n$, with symplectic form ω. Given any function H on $\mathcal{M}$, there is a one parameter group of diffeomorphism given by the integral curves of the vector field V_H defined by

$$\omega(V_H, x) = dH(x), \ \forall x \in T\mathcal{M} \tag{4.21}$$

where H is an hamiltonian function. If the orbits are closed, we have a $U(1)$ action: an hamiltonian action. As we know from classical mechanics, hamiltonian actions preserve the symplectic form and so they are symplectomorphisms. Moreover, this hamiltonian action becomes an isometry if the manifold has also a Kähler structure. Let us consider the action $U(1)^n \times \mathcal{M} \to \mathcal{M}$, this is said to be hamiltonian if its restriction to any $U(1) \subset U(1)^n$ is hamiltonian and any two of them commute. It can be shown that this implies the existence of the so-called moment map $\mu : \mathcal{M} \to \mathbb{R}^n$ whose components are the hamiltonians of each $U(1)$ action. This is the anologue of the map ϕ of the previous chapter. Since all the $U(1)$ commute, for any $\vec{r} \in \mathbb{R}^n$, its preimage $\mu^{-1}(\vec{r})$ is invariant under the action of the full $U(1)^n$. Moreover, since we consider a $\mathcal{M}$ with a Kähler structure, the existence of the hamiltonian action implies that the isometry group contains the algebraic torus $U(1)^n \simeq (\mathbb{C}^*)^n$. So a toric variety emerges in a simple way as a real $2n$ dimensional symplectic manifold $\mathcal{M}$ that has an hamiltonian action of the algebraic torus on it[5].
We are interested in non compact toric varieties but let us talk a little about compact ones. If $\mathcal{M}$ is compact, Delzant [94] showed that the image through the moment map of the variety, $\mu(\mathcal{M})$, is a convex polytope Δ called Delzant polytope

$$\Delta = \{\vec{r} \in \mathbb{R}^n | \vec{r} \cdot \vec{v}_i \le c_i, \vec{v}_i \in \mathbb{R}^n, c_i \ constants\}, \tag{4.22}$$

satisfying the following requests:

- rationality: the vectors $\vec{v}_i$ have integer coordinates;
- simplicity: in each vertex exactly n edges meet.
- smoothness: the edges meeting in a vertex are parallel to integer vectors $\vec{g}_j$ with $j = 1, .., n$ generating the full lattice $\mathbb{Z}^n$.

It is possible to show that, thanks to the Delzant polytope, we can though the compact toric variety as a $U(1)^n$ fibration over Δ; however this is not a usual fibration since turns out that one $U(1) \subset U(1)^n$ shrinks on the edges and so acts

[5]There is a little caveat: if $\mathcal{M}$ is a cone the algebraic torus action must commute with the homothetic action induced by the Euler vector field.

trivially; moreover, since in a vertex n edges meet each others the full $U(1)^n$, fiber shrinks. Hence the vertex of a Delzant polytope are the fixed points of the algebraic torus action $U(1)^n \simeq (\mathbb{C}^*)^n$. The interesting fact is that the vectors $\vec{v}_i$ are normal to the bounding hyperplanes, called also in this case facets, of Δ and turns out that these vectors generate a fan Σ. We refer to this fan as the dual graph of Δ. So at every fan is associated a Delzant polytope and since a fan can describe a compact toric variety we get that every Delzant polytope describe a compact toric variety; moreover this map is one to one and this is the so-called Delzant theorem. In Delzant picture the toric variety is constructed as a quotient of $\mathbb{C}^m$ where m is the number of facets of the Delzant polytope $\bar{\Delta}$. Consider the $n \times m$ matrix

$$V_i^k = \begin{pmatrix} v_1^1 & v_2^1 & \dots & v_m^1 \\ v_1^2 & \ddots & \dots & v_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ v_1^n & v_2^n & \dots & v_m^n \end{pmatrix}, \tag{4.23}$$

note that this is exactly the matrix defined in the homogeneous coordinates approach; its kernel for the rank-nullity theorem has dimension $m - n$ so we can choose a basis $Q^a \in \mathbb{Z}^m$ with $a = 1, ..., m - n$. Consider now $\mathbb{C}^m$ with its standard symplectic Kähler form[6], turns out that there is an $U(1)^{m-n}$ hamiltonian action specified by the charge vectors Q^a and the moment map

$$\mu^a(z_1, ..., z_m) = -\frac{1}{2}\sum_{i=1}^{m} Q^a|z_i|^2 - \xi^a; \tag{4.24}$$

where ξ^a are constants, hence we can consider the quotient of μ^{-1} by the $U(1)^{m-n}$ since μ^{-1} is a subset of $\mathbb{C}^m$ that satisfies 4.24. Putting $\xi^a = 0\ \forall a$ we obtain the so-called symplectic or Kähler quotient

$$\mathbb{C}^m//U(1)^{m-n}; \tag{4.25}$$

since μ^{-1} has real dimension $2m - (m - n) = m + n$ the Kähler quotient has real dimension $2n$ and so complex one n: this is an n-dimensional variety with an algebraic torus action $U(1)^n \simeq (\mathbb{C}^*)^n$, so it is a toric variety and its Delzant polytope is exactly $\bar{\Delta}$. The interesting point is that this construction is well known by physicists: this is the moduli space of the GLSM. It is a SUSY gauge theory in two dimensions with abelian gauge group $U(1)^{m-n}$ and m chiral superfields $z_1, ..., z_m$ with charges $Q^1, ..., Q^{m-n}$. Nevertheless, in a GLSM the charges must sum to zero and this is no the case for a compact toric variety, but we know that this happens for a CY toric cone. So Delzant construction must be generalized to consider non compact toric varieties and this is due to Lerman [95]. In this Delzant-like construction the image under the moment map of $\mathcal{M}$ is no longer a polytope but a cone Θ

$$\Theta = \{\vec{r} \in \mathbb{R}^n | \vec{r} \cdot \vec{v}_i \leq 0, \vec{v}_i \in \mathbb{Z}^n\}, \tag{4.26}$$

and its dual graph is still a fan generated by the normal vectors $\vec{v}_i$; the difference is that this vectors do not generate $\mathbb{R}^n$: this is the fan condition to be non compact also

[6] This is $\omega = \frac{i}{2}\sum_{i=1}^{m} dz_i \wedge d\bar{z}_i$

introduced in the previous section. As we have said before the CY condition impose that $\vec{v}_i$ are coplanar so we can project out the common component and obtain an $n-1$-dimensional object that encode the geometry called toric diagram. Since the components of the vectors $\vec{v}_i$ are integer, the toric diagram is the convex hull of a set of point in a $\mathbb{Z}^{n-1}$-lattice. Lerman [95] showed that the construction of $\mathcal{M}$ from the toric diagram in terms of Kähler quotient, or equivalently in terms of GLSM fields, still holds for toric CY cones and we can still though it as a $U(1)^n$ fibration over the cone Θ. As before this fibration shrinks partially on the edges and shrinks completely on the vertex. Moreover, as we project the fan on a hyperplane we can also project the cone Θ on a hyperplane by simply neglecting the last coordinate; doing this we obtain the so-called web diagram. This web diagram can be constructed taking the line orthogonal to the facet of the triangulated toric diagram.

In the case of our interest the toric diagram and the web diagram are a two dimensional objects and the CY cone threefold exhibits a three dimensional algebraic torus action. In the following we will give same examples of toric diagram and with this pretext we will introduce some new concepts and tools.

Example 1: $\mathbb{C}^3$

Consider the three dimensional fan given by the vectors $\vec{v}_1 = (0,0,1), \vec{v}_1 = (0,1,1), \vec{v}_1 = (1,0,1)$, projecting out the third component we get $\vec{w}_1 = (0,0), \vec{w}_1 = (0,1), \vec{w}_1 = (1,0)$ and the toric diagram is

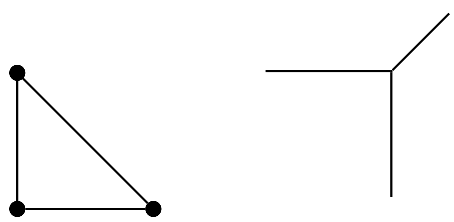

Figure 4.5. *Toric diagram and web diagram of* $\mathbb{C}^3$.

We have $m - n = 3 - 3 = 0$ charge vectors and so nothing to quotient out: the algebraic torus act trivially, this is $\mathbb{C}^3$.

Example 2: $\frac{\mathbb{C}^2}{\mathbb{Z}_2} \times \mathbb{C}$

Consider the fan generated by the vectors $\vec{v}_1 = (0,0,1), \vec{v}_2 = (0,1,1), \vec{v}_3 = (1,0,1), \vec{v}_4 = (-1,0,1)$, projecting out the third component we get $\vec{w}_1 = (0,0), \vec{w}_2 = (0,1), \vec{w}_3 = (1,0), \vec{w}_4 = (-1,0)$ and the toric diagram is

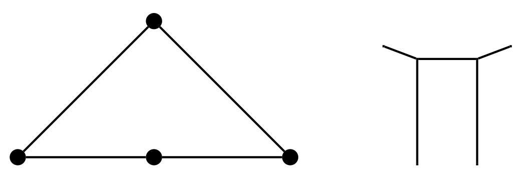

Figure 4.6. *Toric diagram and web diagram of* $\frac{\mathbb{C}^2}{\mathbb{Z}_2} \times \mathbb{C}$.

We have $m - n = 4 - 3 = 1$ charge vector given by

$$\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} = \vec{0} \Rightarrow \begin{cases} Q_3 - Q_4 & = 0 \\ Q_2 & = 0 \\ \sum_{i=1}^{4} Q_i & = 0 \end{cases} \tag{4.27}$$

and so $Q = (-2, 0, 1, 1)$. Since we have one vanishing component, $Q_2 = 0$, the coordinate z_2 do not play role and so we will expect a CY toric manifold of the form $X \times \mathbb{C}$ where X is unknown for the moment and $\mathbb{C}$ is the space associated to z_2. To understand which is X we follow the general algorithm. We have the equivalence relation

$$(z_1, z_2, z_3, z_4) \sim (\lambda^{-2} z_1, z_2, \lambda z_3, \lambda z_4), \tag{4.28}$$

since on z_2 the algebraic torus action is trivial we not consider it and so we must find $m - 1 = 3$ $\mathbb{C}^*$-invariant polynomials, for example

$$x_1 = z_1 z_3 z_4, \; x_2 = z_1 z_3^2, \; x_3 = z_1 z_4^2, \tag{4.29}$$

and they satisfy the relation

$$x_2 x_3 = x_1^2. \tag{4.30}$$

turns out that this is the realization[7] of $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$ as subvariety of $\mathbb{C}^3$. So remember that we must multiply for the $\mathbb{C}$ of z_2 we obtain the toric CY variety $\frac{\mathbb{C}^2}{\mathbb{Z}_2} \times \mathbb{C}$.
This is a general feature: if the set of charge vectors have one or more common vanishing components, the coordinates associated to these charges are transformed trivially by the algebraic torus action $(\mathbb{C}^*)^{m-n}$ and we can write the toric manifold as $X \times \mathbb{C}^k$ where k is the number of common vanishing components of the charge vectors.

Example 3: Suspended Pinched Point (SPP)

Consider the fan generated by the vectors $\vec{v}_1 = (0,0,1), \vec{v}_2 = (0,1,1), \vec{v}_3 = (1,0,1), \vec{v}_4 = (1,1,1), \vec{v}_5 = (0,2,1)$, projecting out the third component we get $\vec{w}_1 = (0,0), \vec{w}_2 = (0,1), \vec{w}_3 = (1,0), \vec{w}_4 = (1,1), \vec{w}_5 = (0,2)$, and the toric diagram is

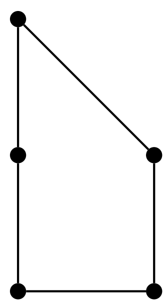

Figure 4.7. *Toric diagram of SPP.*

[7]The $\mathbb{Z}_2$ action can be understand looking at the relation 4.30. We note that if $x_1 \to -x_1$ and $x_2 \to -x_2$ nothing change; this sign flip can be implemented as $z_1 \to -z_1$ and $z_5 \to -z_5$; this is a $\mathbb{Z}_2$ action acting on $\mathbb{C}^2$.

We have $m - n = 5 - 3 = 2$ charge vectors given by

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} Q_1^a \\ Q_2^a \\ Q_3^a \\ Q_4^a \\ Q_5^a \end{pmatrix} = \vec{0} \Rightarrow \begin{cases} Q_3^a + Q_4^a & = 0 \\ Q_2^a + Q_4^a + 2Q_5^a & = 0 \\ \sum_{i=1}^5 Q_i^a & = 0 \end{cases} \tag{4.31}$$

and so $Q^1 = (-1, 1, 1, -1, 0)$ and $Q^2 = (0, 1, -1, 1, -1)$ are solutions and we have the two equivalence relations

$$\begin{aligned} (z_1, z_2, z_3, z_4, z_5) &\sim (\lambda^{-1} z_1, \lambda z_2, \lambda z_3, \lambda^{-1} z_4, z_5) \\ (z_1, z_2, z_3, z_4, z_5) &\sim (z_1, \lambda z_2, \lambda^{-1} z_3, \lambda z_4, \lambda^{-1} z_5). \end{aligned} \tag{4.32}$$

Now we have to find a basis of $m = 5$ $(\mathbb{C}^*)^2$-invariant polynomials; this is, for example

$$x_1 = z_1 z_2 z_5,\ x_2 = z_1 z_2 z_3 z_4 z_5,\ x_3 = z_3 z_4,\ x_4 = z_1^2 z_2 z_3,\ x_5 = z_2 z_4 z_5^2 \tag{4.33}$$

and they satisfy the relations

$$\begin{cases} x_1 x_3 = x_2 \\ x_1 x_2 = x_4 x_5. \end{cases} \tag{4.34}$$

This is the realization as subvariety of $\mathbb{C}^5$ of the so-called Suspended Pinched Point (SPP) toric CY variety.
In both examples 2 and 3 turns out that the toric variety admits a singular line and this behavior can be read directly by the toric diagram: if a diagram has an edge with more than one segment the toric variety described has a singular line. These varieties are called not isolated singularities.

Example 4: del Pezzo and Pseudo del Pezzo surfaces

We introduce the so-called del Pezzo and pseudo del Pezzo surfaces; roughly speaking, these are blow-ups of $\mathbb{CP}^2$. One can ask why we consider blow-ups of a compact variety if we are interested in CY threefold. The fact is that the cone over this surfaces is a CY one and so studying and classifying del Pezzo and pseudo del Pezzo surfaces we obtain potentially new CY cones that can be used to extend AdS/CFT correspondence. Since the projective plane is compact we can use the Delzant polytope technology and in this picture the blow-up operation corresponds to removing one vertex and introducing a facet paying attention that the result is still a Delzant polytope. This is the only way to perform a blow-up on a toric manifold preserving the toric condition. In the dual picture of fan, a blow-ups correspond to add one dimensional cones. The k'th del Pezzo surface (dP_k) is the blow-up of the projective plane $\mathbb{CP}^2$ in k generic points. It is known that del Pezzo surfaces exist for $k = 0, 1, ..., 8$ but for $k > 3$ they are not toric. Below is represented the blow-up sequence from $\mathbb{CP}^2 = \mathrm{dP}_0$ to dP_3 both on Delzant polytopes and fans

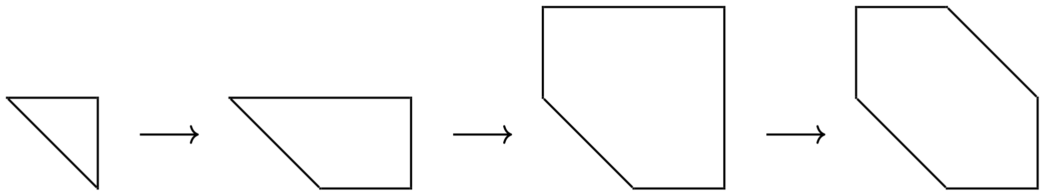

Figure 4.8. *Blow-up sequence $dP_0 \to dP_1 \to dP_2 \to dP_3$ for the toric del Pezzo surfaces implemented on the Delzant polytopes.*

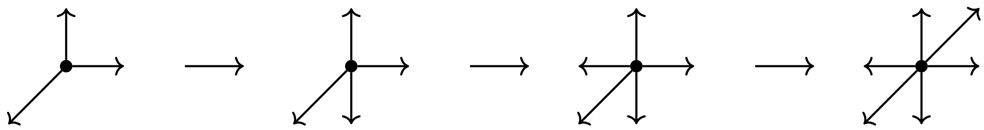

Figure 4.9. *Blow-up sequence $dP_0 \to dP_1 \to dP_2 \to dP_3$ for the toric del Pezzo surfaces implemented on the fans.*

For each toric del Pezzo surface we can construct the cone over it which is a CY cone; the toric diagram of these CY cones over del Pezzo surfaces are represented below: they are the convex hulls of the fan reported in Figure 4.9

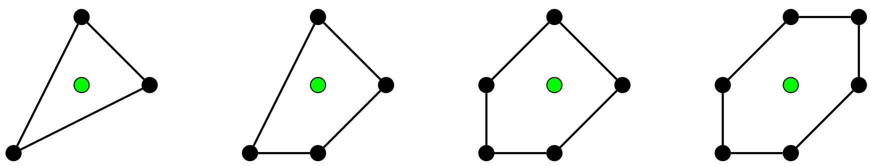

Figure 4.10. *Toric diagrams for the toric CY cone build up over the del Pezzo surfaces. From left to right we have $C(dP_0), C(dP_1), C(dP_2), C(dP_3)$. The origin is represented by a green dot.*

We said that a del Pezzo surface is a blow-up of a generic point[8] of $\mathbb{CP}^2$ but it is possible to blow-up also in non generic points and this non generical blow-up when are toric are named Pseudo del Pezzo (PdP) surfaces. As for del Pezzo surfaces, when we consider the cone over PdP surfaces we get toric CY cone. In Figure 4.11 is represented the possible non generic blow-up of dP_1 that generate PdP_2 both in Delzant polytopes and fans picture.

As we see, the toric diagram of PdP_2 contains an edge with more that one segment: del Pezzo surfaces are isolated singularities while Pesudo del Pezzo are non isolated ones.

[8]Recall that k points in $\mathbb{CP}^2$ are said to be in generic position if there is no line passing through 3 points, no conic passing through 6 points, and no singular cubic passing through 8 points.

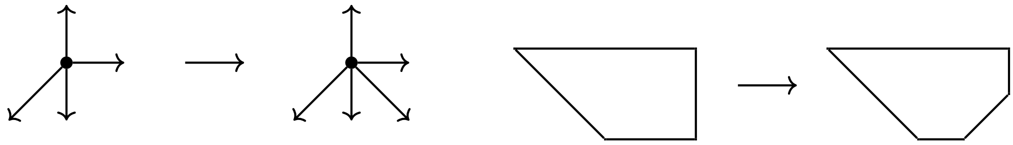

Figure 4.11. *Non generical blow-up for dP_1, this generate PdP_2. Left: blow-up in fans picture. Right: blow-up in Delzant polytopes picture.*

Example 5: zero'th Hirzebruch surface $\mathbb{F}_0$

We saw that it is possible thatna cone over a compact surface is a CY one; another example of this is the so-called zero'th Hirzebruch surface[9]. This is nothing but the surface specified by the left fan of Figure 4.2: it is $\mathbb{CP}^1\times\mathbb{CP}^1$. Its cone, $C(\mathbb{CP}^1\times\mathbb{CP}^1)$, is a toric CY cone. The toric diagram is drown below

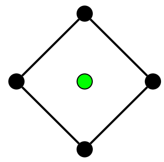

Figure 4.12. *Toric diagram for the cone over the zero'th Hirzebruch surface $\mathbb{F}_0$.*

4.1.3 Reflexive polygons

Let us now analyze an important class of toric diagrams: the so-called reflexive polytopes. We are particularly interested in the two dimensional polytopes (polygons) since we are interested in toric CY threefold.
Reflexive polytopes caught the attention of stringist because they are linked with mirror symmetry. This symmetry is between to CY manifold with Hodge numbers $h^{1,1}$ and $h^{2,1}$ exchanged, hence the understanding of mirror symmetry is to search mirror paired CY manifold. Batyrev and Borisov [97],[98] proposed to search these CY pairs by formulating the construction of CY manifold as hypersurface in toric variety represented by a reflexive polytope toric diagram. The main purpose of this paragraph is to give the classification of all the two dimensional reflexive polytopes and the toric variety associated to them. Before this, let us define what a reflexive polytopes is and give some notions about them.
A reflexive polytope Δ is a convex polytope with points in a $\mathbb{Z}^{n-1}$-lattice[10]that contains a unique interior point; for every reflexive polytope there exist the dual polytope called polar polytope and defined as

$$\Delta^\vee := \{\vec{x}_i \in \mathbb{Z}^{n-1} | \vec{x}_i \cdot \vec{y}_i \geq -1 \ \forall \vec{y}_i \in \Delta\}, \tag{4.35}$$

[9]In general a Hirzebruch surface $\mathbb{F}_q$ is given by $\mathbb{F}_q = \frac{(\mathbb{C}^2\setminus\{0,0\})\times(\mathbb{C}^2\setminus\{0,0\})}{(\mathbb{C}^*)^2} \simeq \frac{\mathbb{C}^4\setminus((\{0,0\},\mathbb{C}^2)\cup(\mathbb{C}^2,\{0,0\}))}{(\mathbb{C}^*)^2}$ with the equivalence relations $(z_1,z_2,z_3,z_4)_1 \sim (\mu z_1, \mu z_2, z_3, z_4)$ and $(z_1,z_2,z_3,z_4)_2 \sim (z_1, z_2, \mu^q \nu z_3, \nu z_4)$. We see that for $q=0$ this is exactly the variety defined by the right fan of Figure 4.2: $\mathbb{CP}^1\times\mathbb{CP}^1$. For more details on Hirzebruch surfaces see [96] chapter 2 paragraph 1.

[10]We use $n-1$ instead of n for coherence with the previous paraghraph: a n-dimensional CY toric variety has a $n-1$-dimensional toric diagram.

this is a true duality relation and so $(\Delta^\vee)^\vee = \Delta$. Turns out that the dual of every reflexive polygon is another reflexive polygon. A reflexive polygon can be self dual. There is a theorem for which in fixed dimension there are only finite many reflexive polytopes up to isomorphisms; however greater is the dimension greater is the number of different reflexive polytopes. Kreuzer and Skarke [99] built up the milestone for a computational algorithm able to classify the reflexive polytopes in fixed dimension up to four dimensions:

- $n-1=2$: we have 16 inequivalent reflexive polygons;
- $n-1=3$: we have 4319 inequivalent reflexive polytopes;
- $n-1=4$: we have 473800776 inequivalent reflexive polytopes;

hence although they are finite, they are in a huge number. Observing the reflexive polytopes in the database it is possible to find the maximal number of vertices of an $n-1$-dimensional reflexive polytope: these are six in two dimensions; fourteen in three dimensions and thirtysix in four dimensions. In the same way, looking at the numbers of vertices for an $n-1$-dimensional simplicial reflexive polytopes one finds: six in two dimensions; eight in three dimensions and twelve in four dimensions. These information are reported in the following table where $\mathcal{V}_M(pol)$ indicates the maximal number of vertices.

	$n-1=2$	$n-1=3$	$n-1=4$
# of reflexive polytopes	16	4319	473800776
$\mathcal{V}_M(pol)$ for reflexive polytopes	6	14	36
$\mathcal{V}_M(pol)$ for simplicial reflexive polytopes	6	8	12

Table 4.1. *Table summarizing some interesting numbers about reflexive polytopes.*

Let be $b=\{\vec{w}_1,...,\vec{w}_k\}$ a set of basis vectors for the projected fan, in other words they are the vectors with which all the vectors of the projected fan can be written; let P the reflexive polytope generated as a convex hull of the projected fan. Let us give some definitions:

- a reflexive polytope is called centrally symmetric if $P=-P$;
- a reflexive polytope is called del Pezzo polytope if k is even and if P is the convex hull of the point indicated by the vectors $(\pm\vec{w}_1,...,\pm\vec{w}_k,\pm(\vec{w}_1+...+\vec{w}_k))$;
- a reflexive polytope is called Pseudo del Pezzo polytope if k is even and if P is the convex hull of the point indicated by the vectors $(\pm\vec{w}_1,...,\pm\vec{w}_k,-(\vec{w}_1+...+\vec{w}_k))$
- a reflexive polytope is called facet symmetric if given a facet F belonging to the reflexive polytope also the facet $-F$ belongs to the reflexive polytope.

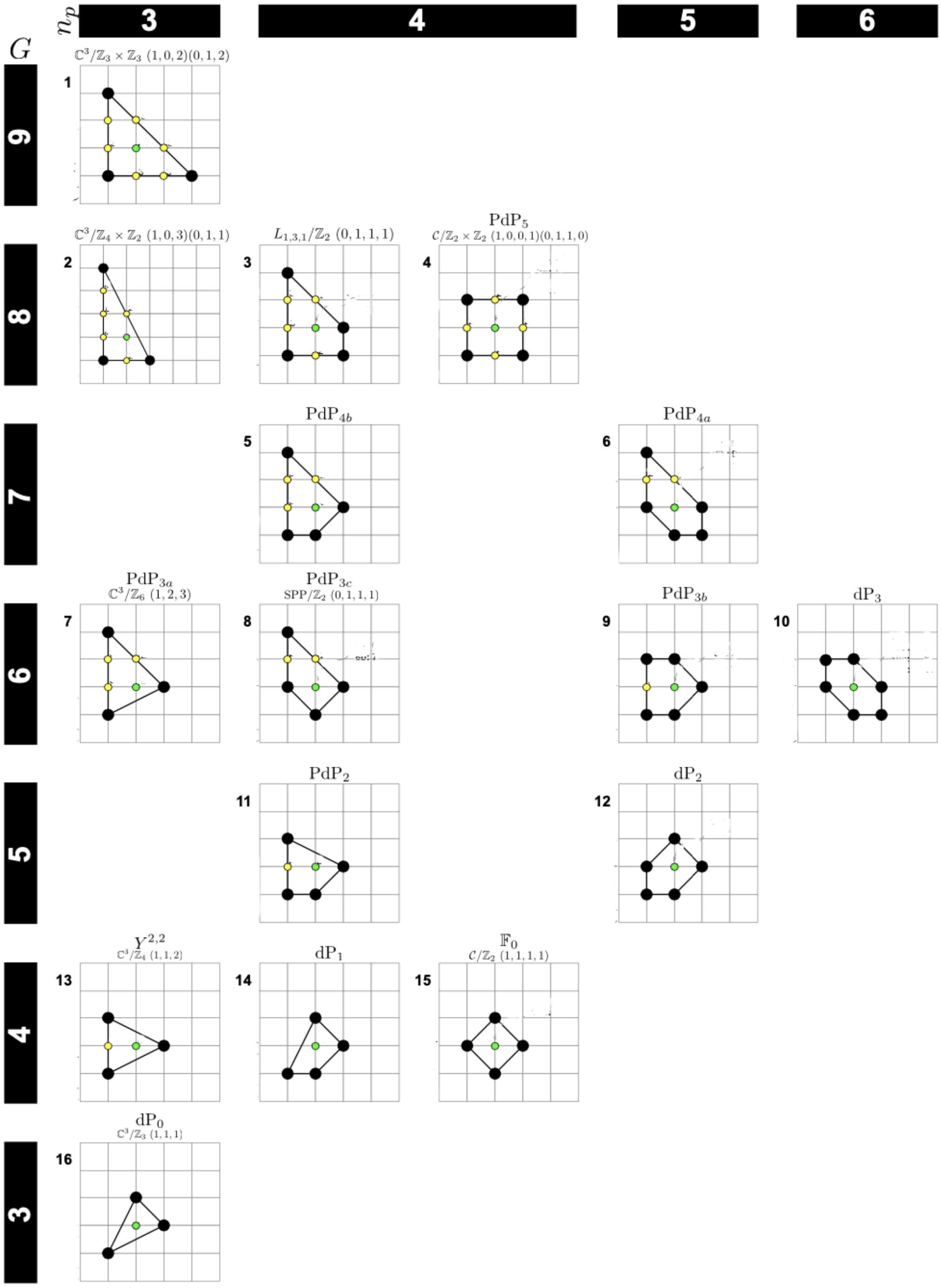

Figure 4.13. *Complete classification of the 16 reflexive polygons; these are the toric diagrams describing toric CY varieties. Green points indicate the origin, this is the only internal point since these are reflexive polygons. Yellow points indicate points that are not vertices while black points are the vertex ones. Vertex points are enumarate horizontally by n_p while the area of the polygons is given vertically by G. The area is calculated considering the smallest lattice triangle (equivalently the toric diagram of $\mathbb{C}^3$) having area $G = 1$. Above every reflexive polygon toric diagram is indicated or the CY variety described or the surface over which constuct the CY threefold or both. We note that, according to point one and two of the previous list, the maximal number of vertices is six. Figure taken and modified from [85]*

Looking at Table 4.1 two conjectures can be made:

1. for an $(n-1)$-dimensional reflexive polytopes $\mathcal{V}_M(pol) \leq 6^{\frac{n-1}{2}}$;
2. for an $(n-1)$-dimensional simplicial reflexive polytopes $\mathcal{V}_M(pol) \leq 3(n-1)$;

the second one was demonstrated by Casagrande [100] while the first one was only proved for centrally symmetric simple[11] reflexive polytopes. We are now ready to specialize at the two dimensional case giving, in the Figure 4.13 above, the entire classification of reflexive polygons.

4.2 Brane tilings and quivers

In previous section we have seen how to encode geometrical information about toric CY cone on a diagram; this geometry information are those that specify the gauge field theory living on the branes that are sitting on the conical singularity of the CY cone. Hence, there is a way to translate the information of toric diagrams in field theory information: matter content, gauge groups and the superpotential. This algorithm allow us to write down the quiver diagram of the theory, an oriented graph with nodes and arrows, and it passes through the so-called brane tiling: a configuration of D5-branes and NS5-branes. Brane tilings and quivers are two ways to encode information on the gauge field theory side of the AdS/CFT correspondence.

4.2.1 Physical interpretation and construction of brane tilings

We consider type IIB superstring theory and a stack of N D5-branes. As we known, we have an $SU(N)$[12] gauge theory living on the D5-branes. However, D5-branes, are six dimensional objects but we want a four dimensional gauge theory; hence two of six directions of D5-branes are redundant and we must compactify them on a torus of radius R, $\mathbb{T}^2$. Conventionally, from the ten dimensional space-time coordinates $X^0, X^1, ..., X^9$ we take X^5 and X^7 to be directions of the torus. We refer to these coordinates simply as 5 and 7. So, now the picture is a stack of N D5-branes wrapping the torus $\mathbb{T}^2$. If R is small and mass modes decouple, we have, thanks to AdS/CFT correspondence, an effective four dimensional $\mathcal{N} = 4$ SYM theory. However, as usual, we want to reduce supersymmetry down to $\mathcal{N} = 1$; for those purposes we have to add another ingredient: NS5-branes. We divide the D5-branes worldvolume intersecting it with NS5-branes; the result are more D5-branes worldvolume regions, each of them having its own gauge group $SU(N)$. The resulting gauge theory has gauge group given by $\prod_{i=1}^{k} SU_k(N)$ where k is the number of D5-branes worldvolume sections. Moreover, if the NS5-branes are introduced in two different directions, only a quarter of the initial supercharges is preserved: this is a $\mathcal{N} = 1$ SYM theory. This D5-branes and NS5-branes picture is represented in the following figure.

[11] An $(n-1)$-dimensional simple polytope is an $(n-1)$-dimensional polytope each of whose vertices are adjacent to exactly $(n-1)$ edges or, equivalently, to $n-1$ facets. This means that the vertex figure (the figure exposed when a corner is sliced off) of a simple $(n-1)$-dimensional polytope is a $(n-2)$-simplex.

[12] Since the $U(1)$ decouples in IR, we do not bother the difference between $U(N)$ and $SU(N)$

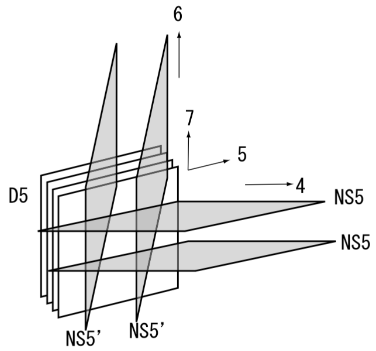

Figure 4.14. *Fivebrane system example. This configuration provides a four dimensional $\mathcal{N} = 1$ SYM theory with gauge group $\prod_{i=1}^{k} SU_k(N)$. This specific configuration is the conifold. Figure taken from [92].*

As we can see from Figure 4.14, we have junctions between NS5-branes and the stack of D5-branes; in order to avoid problems with charge conservation [102] the junction should be as shown in the Figure 4.15.

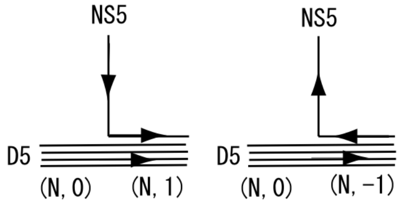

Figure 4.15. *Allowed junction configurations for the fivebrane system. The parentheses at the bottom label the NS5-branes D_5-branes bound states. Figure taken from [92].*

We see that some regions of the torus becomes bound states of N D5-branes and one NS5-brane. In general, bound states of N D5-branes and n NS5-branes are called $(N,\, n)$-branes. In the case $n < 0$, $(N,\, n)$-brane is a bound state of N D5 and $|n|$ NS5-branes, with D5-branes and NS5-branes having opposite orientations. This explains the meaning of the parentheses at the bottom of Figure 4.15. Such a division of $\mathbb{T}^2$ view on a plane with boundary identification is called fivebrane diagrams since they represent the structure of fivebrane systems.

Two commentes are in order now. First, due to the junction condition of Figure 4.15, we started with NS5-branes orthogonal to D5-branes and we end with NS5-branes parallel to D5-branes: these initially different NS5-branes join together and we have a single NS5-brane; however, we must remember that the orientation of the NS5-brane is opposite in some regions. Note that the shape of this NS5-brane is singular since it curves of ninety degrees; this is because we are, implicitly, considering the strong coupling limit in which the brane configuration simplifies. The shape of NS5-brane is obviously smooth for general string coupling constant, and in the weak coupling limit becomes a holomorphic curve.

Second, let us call x-cycle and y-cycle the two cycles of the torus and since we are on a torus, the NS5-brane charge should be the same after we go around an arbitrary cycle of $\mathbb{T}^2$; this means that labelling with p_i and q_i the winding number referred to x-cycle and y-cycle of the torus of an arbitrary NS5-branes cycle, we must have the condition $\sum_i p_i = \sum_i q_i = 0$ where i runs over the number of NS5-branes cycles. This implies that not all the fivebrane diagram configurations are allowed; we can have only fivebrane diagrams in which the winding numbers are summed up to zero.

It is now time to understand the connection between the fivebrane system and what we are interested in, D3-branes sitting in a conical singularity of some toric CY threefold cone. The connection is simply T-duality: since directions 5 and 7 are $\mathbb{T}^2$ we can T-dualize in these directions and D5-branes are turned into D3-branes while NS5-brane are turned into CY threefold geometry[13]. From D3-brane picture, the torus $\mathbb{T}^2$ are subtorus of the $U(1)^3 \simeq (\mathbb{C}^*)^3$ isometry of toric Calabi-Yau threefold, and we have taken T-duality along that $\mathbb{T}^2$ to turn Calabi-Yau geometry into NS5-brane. Thanks to Buscher's rule [103], turns out that the web diagram coincides with junction of NS5-branes. Hence, now we have an algorithm to write down the fivebrane diagram of a toric diagram, or in other words, to find the fivebrane configuration dual to the D3-branes sitting in a CY conical singularity picture. Given a toric diagram the algorithm is the following:

1. draw the web diagram of the triangulated toric diagram;

2. in a topological torus (a square with boundary identification) draw the NS5-brane cycles corresponding to the edges of the web diagram that are orthogonal to extremal edges of the triangulated toric diagram. Pay attention to respect the condition that winding numbers must sum to zero $\sum_i p_i = \sum_i q_i = 0$.

Doing this fivebrane diagrams we note that we have polygons where starting from a point of the perimeter we can return to the starting point following the NS5-brane cycles; on the other hand we have also polygons for which this is not possible. Noting that we can construct the bipartite graph in the following way:

1. we enumerate the inequivalent polygonal faces of our fivebrane diagram for which is not possible to come back to starting point following the NS5-branes cycles;

2. we draw a black circle inside every polygonal face that has as boundary closed counterclockwise curve given by the NS5-brane cycles;

3. we draw a white circle inside every polygonal face that has as boundary closed clockwise curve given by the NS5-brane cycles;

4. we draw little arrows between all the enumerated polygonal faces that have shared vertex in such a way that following this arrows we go clockwise around white circles and counterclockwise around black circles.

[13]Recall that T-duality exchanges momentum and winding. This means that corresponding gauge fields (metric and KR field), should also be exchanged. In the original Calabi-Yau picture, we have no NS-NS KR field but the metric is not flat. After T-duality, we have a non trivial KR field which is the source of NS5-brane and a flat geometry.

Let us understand better the physical interpretation of bipartite graph. The enumerated polygonal faces correspond to those part of the stack of N D5-branes on which there is no bounded NS5-brane: (N,0) and these are $SU(N)$ gauge groups according to what we said at the beginning of this paragraph. In the same way, the black and white circles (or equivalently, the polygonal faces with a fixed direction NS5-branes cycles as boundary) correspond, respectively and in a conventional manner, to (N, 1) and (N, −1) bound states. These correspond to not dynamical $U(1)$ global groups [104],[105]. Last but not least, the arrows between enumerated polygonal faces that have shared vertex are open strings attached to the different sections of the stack of N D5-branes. These open strings contribute with bifundamental fields in the worldwolume SYM field theory that are in the fundamental representation of the gauge group corresponding to the enumerated polygonal face of departure of the arrow and in the antifundamental representation of the gauge group corresponding to the enumerated polygonal face of arrival of the arrow. We will indicate this fields as X_{da} where d corresponds to the number of the departure polygonal face and a to the number of the arrival polygonal face. So a brane tiling contains information about gauge groups and matter content but there is even more: it contains also the superpotential of the SYM theory. Every black and white circle is a tree level disk amplitude interaction of strings which means that we have such a term in the superpotential: for each circle we have a term in the superpotential given by the trace over gauge indexes of the product of fields X_{da} associated to arrows that are around the given circle. The correct order is given following the arrows flow and depending on the color of the circle; this term enters with a minus (white circle) sign or a plus (black circle) sign. This choice of the proportionality constants ensures that the theory becomes conformal and the corresponding moduli space geometry to be a toric Calabi-Yau cone. To summarize:

- to each differently enumerated polygonal face we associate a $SU(N)$ gauge group. Hence the gauge group G of the theory is

$$G = \prod_{i=1}^{k} SU_k(N), \tag{4.36}$$

  where k is the number of differently enumerated polygonal faces;

- to each arrow between two enumerated polygonal face we associate a bifundamental field living in the fundamental representation of the gauge group corresponding to the enumerated polygonal face of departure of the arrow and in the antifundamental representation of the gauge group corresponding to the enumerated polygonal face of arrival of the arrow. There are as many arrows, and so bifundamental fields, as intersections created by NS5-brane cycles;

- to each circle we associate a term in the superpotential given by the trace over gauge indexes of the product of fields X_{da} associated to arrows that are around the given circle. The correct order is given following the arrows flow and, depending on the color of the circle, this term enters with a minus (white circle) sign or a plus (black circle) sign. Hence the superpotential is given by

$$W = \sum_{white\ circles} Tr\Big(\prod_{i=1}^{n} X_{d_i a_i}\Big) - \sum_{black\ circles} Tr\Big(\prod_{i=1}^{n} X_{d_i a_i}\Big), \tag{4.37}$$

where n is the number of arrows around a given circle and the trace in over gauge indexes.

We will see that bipartite graphs contain all the information we need on the SYM gauge theory. This is a quiver theory and since, in the general configuration of D5-branes and NS5-brane of Figure 4.14, only a quarter of the initial thirtytwo supercharges is preserved and the introduction of NS5-brane makes the D5-branes free to move only in two directions, this is a chiral[14] $\mathcal{N} = 1$ superconformal Yang-Mills theory.
However, all the information contained into bipartite graph can be transposed in a new object called dimer; this contains exactly the same information of bipartite graph but has the advantage to be more immediate. To construct it we have to do the following steps:

1. draw on a plane a polygon associated to each (N,0) region. The number of edges is the same of the enumerated polygonal face of the bipartite graph. Enumerate this polygons with the same numbers of the enumerated polygonal face of the bipartite graph.
2. the bound state regions (N, ± 1) are instead the vertices of the polygons. Here bipartiteness manifests itself in the fact that two adjacent vertices always have opposite colors;
3. find a fundamental cell with the property that if we repeat it all over the plane we obtain the entire dimer. The choice of the fundamental cell is not unique.

The information about the field theory are read off from the fundamental cell in a way very similar to previous one for bipartite graph:

- to each differently enumerated polygons correspond a $SU(N)$ gauge group, and so the gauge group of the theory is given by
$$G = \prod_{i=1}^{k} SU_k(N), \tag{4.38}$$
where k is the number of differently enumerated polygons;
- we write an arrow orthogonal to each edge of the enumerated polygons; to each edge we associate a bifundamental field living in the fundamental representation of the gauge group corresponding to the polygon from which the arrow starts and in the antifundamental representation of the gauge group corresponding to the polygon from which the arrow arrives;
- the superpotential are given by
$$W = \sum_{white\ vertex} Tr\Big(\prod_{i=1}^{n} X_{d_i a_i}\Big) - \sum_{black\ vertex} Tr\Big(\prod_{i=1}^{n} X_{d_i a_i}\Big), \tag{4.39}$$
where n is the number of arrows around a given vertex and the trace is over gauge indexes.

As we see dimers and bipartite graphs are completely equivalent.

[14]This is because we have only two scalars: the coordinates of the two directions along which the D5-branes can move.

Example 1: $\mathbb{C}^3$

Let us start with the simplest example: $\mathbb{C}^3$. This is the toric CY cone geometry corresponding to the original AdS/CFT correspondence and we know that the field theory is a $\mathcal{N} = 4$ SYM theory with one gauge group $SU(N)$, three bifundamental fields in the adjoint representation and superpotential $W = Tr(X[Y, Z])$. Let us check this known results.
Consider the toric diagram and the associated web diagram given in Figure 4.5; from these it is simple to write down the bipartite graph and then the dimer.

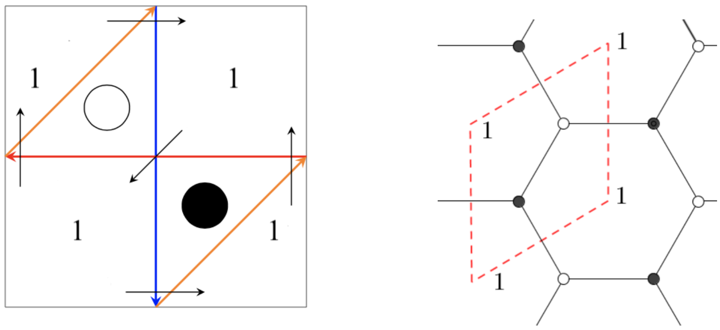

Figure 4.16. *Bipartite graph and dimer for $\mathbb{C}^3$. We have one $SU(N)$ gauge group, three fields and two superpotential terms. In red an example of fundamental cell.*

We have three NS5-brane cycles that we can represent as in the left Figure 4.16; we have two circles and only one enumerated hexagonal face if we remember that we are on a topological torus. We can draw three independent arrows corresponding to three intersections of the NS5-brane cycles; these are three fields that are in the adjoint representation of the only $SU(N)$ gauge group since the arrows start and arrive at the same enumerated polygonal face. These are the fields X, Y and Z: X is the field associated to the transverse arrow, Y is the field associated to the vertical arrows and Z is the field associated to the horizontal arrows. For the superpotential note that there are two terms given by $W_1 = +Tr(XYZ)$ and $W_2 = -Tr(XZY)$; hence

$$W = Tr(X[Y, Z]) \tag{4.40}$$

as expected. The same results can be found looking at the red fundamental cell of the dimer on the right of Figure 4.16.

Example 2: the conifold $\mathcal{C}$

We consider now the conifold case; this is the CY geometry studied by Klebanov and Witten as we have seen at the and of Chapter 3. This corresponds to a SUSY field theory with gauge group $G = SU(N_1) \times SU(N_2)$ and with four bifundamental fields; moreover we know that the superpotential has the known form 3.46, $W = Tr(X^1_{12}X^1_{21}X^2_{12}X^2_{21}) - Tr(X^1_{12}X^2_{12}X^2_{21}X^1_{21})$ where we have relabelled the fields as $X^1_{12} := A_1, X^1_{21} := B_1, X^2_{12} := A_2, X^1_{21} := B_2$. When Klebanov and Witten built up their model, they did not know all the technology of bipartite graphs and dimers, so let us check that this algorithm returns the correct answer.

The toric diagram is in Figure 4.4 and corresponding web diagram leads to four NS5-brane cycles; bipartite graph and dimer for the conifold is reported below

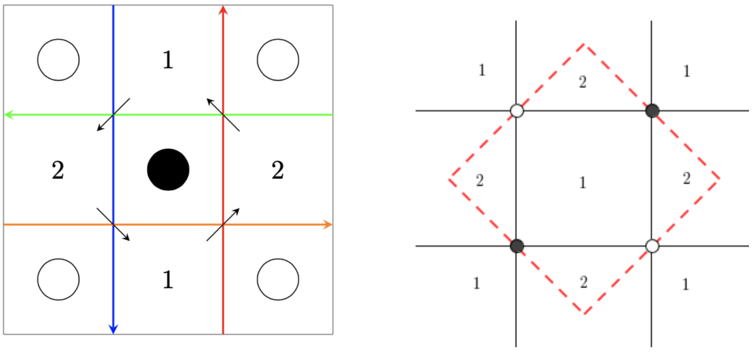

Figure 4.17. *Bipartite graph and dimer for the conifold, $\mathcal{C}$. We have two $SU(N)$ gauge groups, four fields and two superpotential terms. In red an example of fundamental cell.*

We see that we have a gauge group $G = SU(N_1) \times SU(N_2)$ since there are two different square faces and four bifundamental fields $X^1_{12}, X^1_{21}, X^2_{12}, X^1_{21}$, since we have four arrows. Finally, the superpotential is given by two pieces

$$W = Tr(X^1_{12}X^1_{21}X^2_{12}X^2_{21}) - Tr(X^1_{12}X^2_{21}X^2_{12}X^1_{21}) \tag{4.41}$$

as we expected again.

Example 3: SPP

Let us consider SSP toric diagram 4.7 and its web diagram; these allow us to draw bipartite graph and dimer

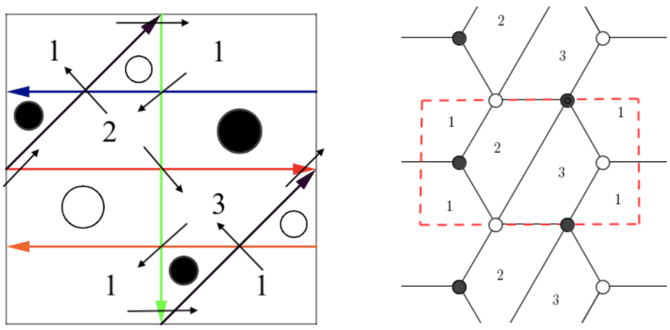

Figure 4.18. *Bipartite graph and dimer for SPP. We have three $SU(N)$ gauge groups, seven fields and four superpotential terms. In red an example of fundamental cell.*

We see that we have a gauge group $G = SU(N_1) \times SU(N_2) \times SU(N_3)$ since there are three different faces: two squares and one hexagon. We have seven bifundamental fields and the superpotential contains four pieces, two quartic and two cubic:

$$W = Tr(X_{23}X_{31}X_{13}X_{32} - X_{12}X_{23}X_{32}X_{21} + X_{12}X_{21}X_{11} - X_{13}X_{31}X_{11}). \tag{4.42}$$

4.2.2 From dimers to quivers

We saw that bipartite graphs or dimers contain the information that we need to specify the field theory that AdS/CFT correspondence associate to a stack of N D3-branes sitting in a conical singularity of toric CY cone. However, it is not easy to see the content of a bipartite graph or a dimer at glance especially when we have theories with many $SU(N)$ gauge groups. So, we want to take the field theory information and represent them in a diagram that is more intuitive: this diagram is the quiver diagram of the theory. Let us give the algorithm to construct it:

1. to each $SU(N)$ gauge group we write a node with a number;
2. to each bifundamental field we associate an arrow pointing to the node corresponding to gauge group under which the field transforms into the antifundamental representation;
3. superpotential terms are closed loops in the quiver diagram, however this is no a one to one correspondence: not all the closed loops are superpotential terms.

Let us give some examples.

Example 1: $\mathbb{C}^3$

We have one gauge group and three adjoint fields so the quiver diagram is

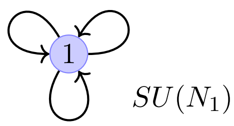

Figure 4.19. *Quiver diagram for the simple example of $\mathbb{C}^3$; this represents a field theory with one $SU(N)$ gauge group and three fields in the adjoint representation.*

Example 2: the conifold $\mathcal{C}$

For the conifold case, we already now the quiver diagram, as it appears in the Klebanov Witten model

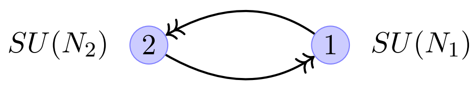

Figure 4.20. *Quiver diagram for Klebanov Witten model: we have two gauge groups and four chiral bifundamental fields.*

Let us use this example to show how a-maximization works and to perform this calculation explicitly in the case $N_1 = N_2 := N$. We recall that a-maximization allows us to find the R-charges of the fields. The superpotential is $W = \epsilon_{ij}\epsilon_{kl}Tr\big(X^i_{12}X^k_{21}X^j_{12}X^l_{21}\big)$ and it must have R-charge two; we note that there is a $SU(2) \times SU(2)$ symmetry that implies $R[X^1_{12}] = R[X^2_{12}]$ and $R[X^1_{21}] = R[X^2_{21}]$. Recall that the R-charge of a fermion is $r = R - 1$ where R is the R-charge of the scalar component of the superfield. From the require that $R[W] = 2$ we get

$$2 = 2R_{12} + 2R_{21} = 2r_{12} + 2r_{21} + 4 \Rightarrow r_{12} + r_{21} = -1; \tag{4.43}$$

moreover, we have to impose that the R-symmetry is not anomalous[15]

$$\begin{cases} 2r_{12}N^2 + 2r_{21}N^2 & = -2N^2 \\ 2r_{12}N^2 + 2r_{21}N^2 & = -2N^2 \end{cases} \tag{4.44}$$

the first equation corresponds to the zero'th gauge group and the second one to the other gauge group. The l.h.s correspond to chiral fermions living in the bifundamental representation and so are $N \times N$ matrices while the r.h.s correspond to gauginis that are $N \times N$ matrices and have conventionally R-charge one. As we can see system 4.44 do not give as new information, so we have only the constrain $r_{12} = -1 - r_{21}$. It is now time to maximize the central charge a:

$$\begin{aligned} a_T &= \frac{3}{32}[3Tr(R^3) - Tr(R)] = \\ &= \frac{3}{32}[3(2N^2r_{12}^3 + 2N^2r_{21}^3 + 2N^2) - 2N^2r_{12} - 2N^2r_{21} - 2N^2] = \\ &= \frac{3}{32}[6N^2(r_{12}^3 + r_{21}^3 + 1) - 2N^2\underbrace{(r_{12} + r_{21} + 1)}_{=0}] = \frac{12N^2}{32}[(-1 - r_{21})^3 + r_{21}^3 + 1], \end{aligned} \tag{4.45}$$

this function is maximized for $r_{21} = -\frac{1}{2}$ and so $r_{12} = -\frac{1}{2}$, as we expected. Note that we have only one variable: this is correct since we have two gauge groups that contain a $U(1)$ but one of these two $U(1)$ subgroups do not play any role in this discussion.

Example 3: abelian orbifolds of $\mathbb{C}^3$

What we call orbifolds are the Calabi-Yau geometry of the form $\frac{\mathbb{C}^3}{\Gamma}$ where Γ is a discrete subgroup of $SU(3)$. The request $\Gamma \subset SU(3)$ is needed to ensure that some supersymmetry is preserved. It is possible to show that when $\Gamma \subset SU(2)$, the CY geometry preserves twice the supersymmetry with respect to the general case: these are in general $\mathcal{N} = 2$ SUSY theories. Since $\Gamma \subset SU(2)$, one of the three complex coordinates of $\mathbb{C}^3$ is untouched by Γ action and the geometry is $\frac{\mathbb{C}^2}{\Gamma} \times \mathbb{C}$ as in the case of $\frac{\mathbb{C}^2}{\mathbb{Z}_2} \times \mathbb{C}$.
When Γ is not contained in $SU(2)$, the gauge theory has minimal supersymmetry: these are in general $\mathcal{N} = 1$ SUSY theories. Moreover, the abelian orbifold groups $\Gamma = \mathbb{Z}_n$ and $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_m$ make the geometry toric and so we focus on them. Let us consider the case $\Gamma = \mathbb{Z}_n$ since generalization for the other is quite straightforward. Let $(z_1, ...z_q)$ the coordinate of $\mathbb{C}^q$, the orbifold group action is given by

$$(z_1, ...z_q) \to \left(e^{\frac{2\pi i k_1}{n}} z_1, ..., e^{\frac{2\pi i k_q}{n}} z_q\right) = (\omega^{k_1} z_1, ..., \omega^{k_q} z_q) \tag{4.46}$$

[15] We choose the anomaly index of the fundamental equal to one and hence the anomaly index of the adjoint is 2.

with $\omega = e^{\frac{2\pi i}{n}}$: we see the analogy with what was discussed in the paragraph 4.1.1 hence seems natural that $k_1, ..., k_q$ must satisfy the CY condition: $\sum_{i=1}^{q} k_i = 0$ but since we are talking about cyclic groups, this condition must be true modulo the order of the group, n. Generally, orbifold action are specified by a vector with q integer components $(k_1, ..., k_q)$. As explicit example let us consider $\frac{\mathbb{C}^2}{\mathbb{Z}_2} \times \mathbb{C}$; toric diagram and web diagram are in Figure 4.6. From web diagram we constuct bipartite graph and dimer as shown in figure below

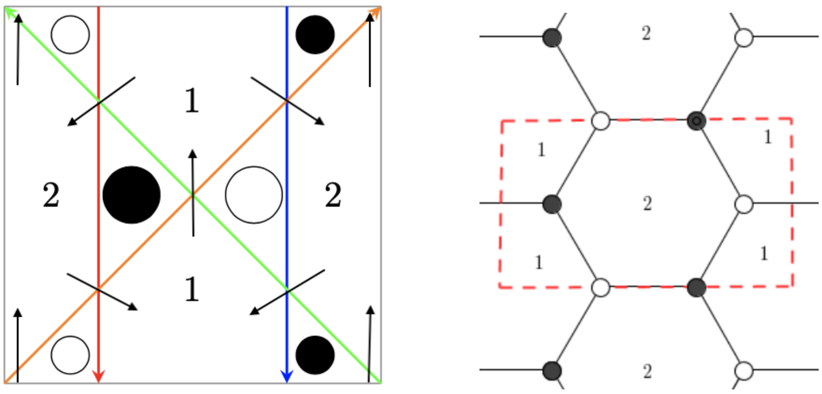

Figure 4.21. *Bipartite graph and dimer for abelian orbifold* $\frac{\mathbb{C}^2}{\mathbb{Z}_2} \times \mathbb{C}$. *We have two* $SU(N)$ *gauge groups, six fields and four superpotential terms. In red an example of fundamental cell.*

We have two $SU(N)$ gauge groups and six bifundamental fields; looking at the white and black circles or vertices we find the superpotential of the theory

$$W = Tr(X_{11}X_{12}^1X_{21}^2) - Tr(X_{11}X_{12}^2X_{21}^1) - Tr(X_{12}^1X_{22}X_{21}^2) + Tr(X_{12}^2X_{22}X_{21}^1). \quad (4.47)$$

The quiver diagram of the theory is

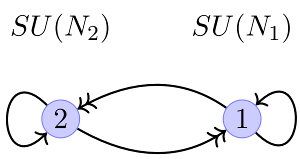

Figure 4.22. *Quiver diagram of the abelian orbifold* $\frac{\mathbb{C}^2}{\mathbb{Z}_2} \times \mathbb{C}$.

There are several features that hold for all abelian orbifold of $\mathbb{C}^3$:

- The dimer contains all hexagons. It was expected since it is true for $\mathbb{C}^3$ and the brane tiling is nothing but a T-duality transposition of the geometry. Therefore, one could guess that the orbifold simply reorganizes the face or gauge group assignment on the dimer; furthermore the superpotential only contains cubic terms;

- for a $\mathbb{Z}_n$ orbifold, we obtain n gauge groups, and $3n$ chiral superfields;
- turns out that all the matter fields have R-charge equal to $\frac{2}{3}$, hence the central charge value is $a = \frac{9}{32}\left(nN^2 + 3nN^2\left(-\frac{1}{3}\right)^3\right) = \frac{nN^2}{4}$.

The fact that all matter fields have R-charge $\frac{2}{3}$ will be clear when we will talk about isoradial embedding in the next paragraph.

Example 4: abelian orbifold of the conifold

As last example let us consider abelian orbifold of the conifold. The toric diagrams are parallelograms and we consider the interesting class of them that have squared toric diagrams, with one edge starting from $(0,0)$ and ending on (k,l) where k and l are coprime numbers. These correspond to $\mathbb{Z}_{k^2+l^2}$ orbifolds of the conifold. In figure below we show three examples of this

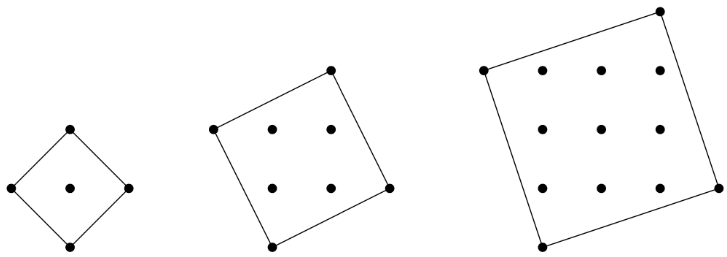

Figure 4.23. *Three examples of conifold's abelian orbifold. From left to right we have orbifold group $\mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_{10}$.*

We note that the first from left is nothing but the transformed $SL(2,\mathbb{Z})$ of the toric diagram of the cone over $\mathbb{F}_0$. If it were not clear, two $(n-1)$-dimensional toric diagrams linked by a $SL(n-1,\mathbb{Z})$ transformation are equivalent; hence the first from left in Figure 4.23 is exactly the toric diagram of the cone over $\mathbb{F}_0$. In the end the cone over $\mathbb{F}_0$ is an abelian orbifold of the conifold. Let us study this with more attention; bipartite graph and dimer is drown in figure below 4.24

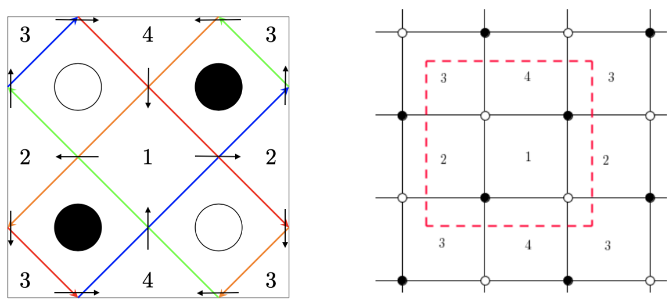

Figure 4.24. *Bipartite graph and dimer for $\mathbb{Z}_2$ abelian orbifold of the conifold or, equivalently, for the cone over $\mathbb{F}_0$. We have four $SU(N)$ gauge groups, eight fields and four quartic superpotential terms. In red one possible choice of fundamental cell.*

We see that we have four $SU(N)$ gauge groups and eight bifundamental fields, the superpotential can be read off simply and it is

$$W = Tr(X_{41}^1 X_{12}^2 X_{23}^1 X_{34}^2 - X_{41}^1 X_{12}^1 X_{23}^1 X_{34}^1 + X_{12}^1 X_{23}^2 X_{34}^1 X_{41}^2 - X_{41}^2 X_{12}^2 X_{23}^2 X_{34}^2) \tag{4.48}$$

The quiver diagram of the theory is given in Figure 4.25.

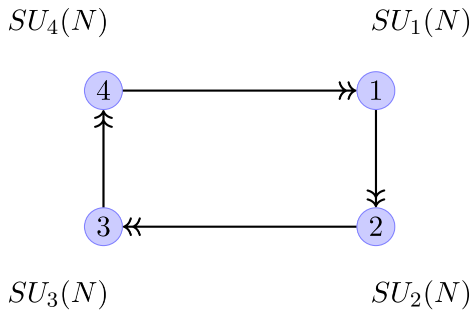

Figure 4.25. *Quiver diagram for the $\mathbb{Z}_2$ orbifold of the conifold.*

Note that in all this examples every field appears only twice in the superpotential, once with a minus sign and once with a plus sign. This is indeed a general property of toric quiver gauge theories, and it is sometimes dubbed as toric condition of superpotential. Another general property that we can note is that there is always an even number of fields in the fundamental and antifundamental representations of a given gauge group; hence, for a given node, the number of outgoing arrows is equal to incoming ones. Infact, to avoid neighborhood vertices with the same color in dimers, each polygon must have an even number of edges.

4.2.3 Some brane tiling and quiver tools

Brane tilings as periodic bipartite graphs on the two torus are computationally far more superior than a quiver and toric superpotential on their own. This is because as a graph, brane tilings owns many graphical properties that can be used as effective tools in the computation of physical quantities of the corresponding superconformal field theory; these "moves" at diagrams level know a lot of physics and give us pictorial methods to understand what happens physically.

Perfect matchings and zig-zag paths

A perfect matching p_j is a set of bifundamental fields which connects to all nodes in the brane tiling precisely once. It corresponds to a point in the toric diagram [106],[107] of the Calabi-Yau threefold. Perfect matchings can be summarized in a matrix $m \times n$ where m is the number of matter fields and n is the number of perfect matchings, P; this matrix is defined as

$$P_{ij} = \begin{cases} 1 & if \ X_i := X_{da} \in p_j \\ 0 & if \ X_i := X_{da} \notin p_j \end{cases}. \tag{4.49}$$

A winding number $w = (p, q)$ can be assigned to an oriented object that passes between two copies of the fundamental cell of a brane tiling and so we can assign it to a perfect matching in the following way:

- we choose, conventionally, a positive direction for the two cycles of $\mathbb{T}^2$;
- we choose an orientation for the edges (and so for fields) of the dimer, for example from white vertex to black one;
- for each edge (field) in a given perfect matching that cross the fundamental cell in the positive x-cycle direction we assign $p = 1$ and $p = -1$ otherwise;
- for each edge (field) in a given perfect matching that cross the fundamental cell in the positive y-cycle direction we assign $q = 1$ and $q = -1$ otherwise;
- we assign the winding number $w(p_j) = \sum_{r=1}^{k} w(X_{d_r a_r})$ where $w(X_{d_r a_r})$ is assigned by the previous rules and k is the number of edges (fields) of the perfect matching.

We can think at winding numbers of all perfect matchings of a brane tiling as $\mathbb{Z}^2$-lattice coordinates of a set of points, then the convex hull of this lattice points forms a polygon which can be identified as the toric diagram of the toric CY threefold corresponding to the dimer[16] [106]. Moreover, thanks to perfect matchings, it is possible to define a new basis of fields from the set of quiver fields in order to describe both F-term and D-term constraints of the supersymmetric gauge theory. These new basis fields are interpreted as GLSM fields and turns out that they precisely correspond to perfect matchings of the brane tiling [106]. An example is shown in the following Figure 4.26: we choose the positive edges orientation from white vertex to black one. The y-cycle is the vertical green dotted line and its positive direction is upward; the x-cycle is the transverse green dotted line and its positive direction is to the left. The matrix P in this case is

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}; \qquad (4.50)$$

where, for example, $P_{11} = 1$ tells us that $X_1 := X_{11}$ is a field contained in p_1 while $P_{13} = 0$ tells us that $X_1 := X_{11}$ is a field not contained in p_3.

Obviously, it can happen that more than one perfect matching has the same winding number; since we have said that a perfect matching identifies a point of the toric diagram, if more perfect matchings have the same $w(p_j)$ they correspond to the same toric diagram's point and we call multiplicity of this point the number of perfect matchings with the same winding number.

[16]Recall that the toric diagram is $SL(2, \mathbb{Z})$ invariant.

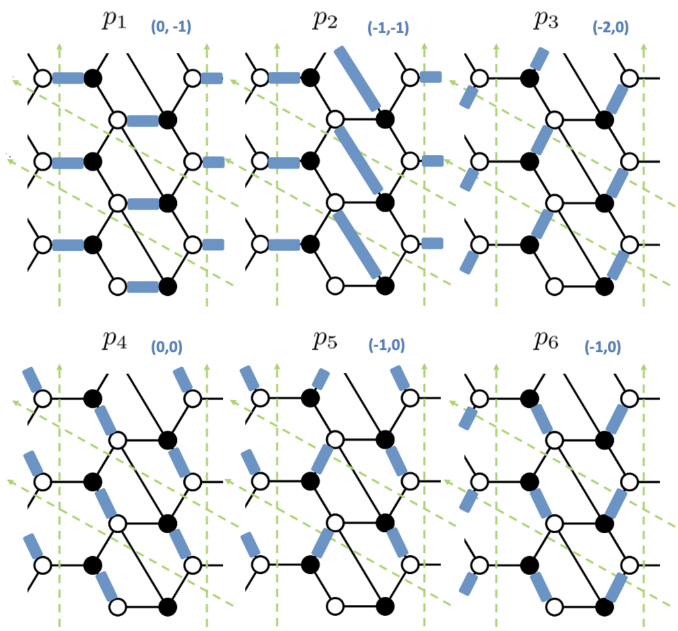

Figure 4.26. *Examples of perfect matchings: these are the six perfect matching of SPP model. We choose the positive edges orientation from white vertex to black one. The y-cycle is the vertical dotted line and its positive direction is upward; the x-cycle is the transverse dotted line and its positive direction is to the left. Winding numbers of the prefect matching are shown at the top right. Figure taken from [85].*

We know that the toric diagram's dual graph is the web diagram and so we expect there is pictorial method analogue to perfect matching that is able to give us information about web diagram: these are the so-called zig-zag paths. A zig-zag path $\tilde{\eta}_j$ is a closed path along the brane tiling which passes between white and black vertices, respects the conventional choice of rotation around them and crosses the entire dimer along one of the two torus cycles. It easy to convince ourselves that zig-zag paths are in one to one correspondence with the NS5-brane cycle that wrap $\mathbb{T}^2$. Infact, every zig-zag path has a winding number in relation to a reference fundamental cell of the brane tiling given following the same rules as before. The winding numbers of the zig-zag paths of a brane tiling can be drawn as rays from the origin of a $\mathbb{Z}^2$-lattice. We call the resulting fan the reduced web diagram and from this we can obtain the web diagram by decomposing the origin into more trivalent vertices. Below in Figure 4.27 the example for SPP model.

We have to highlight that not all zig-zag paths or perfect matchings are allowed; there are some properties they must satisfy in order to have a consistent theory:

- no vertex point of the toric diagram corresponds to more than one perfect matching;
- no zig-zag paths self intersect.

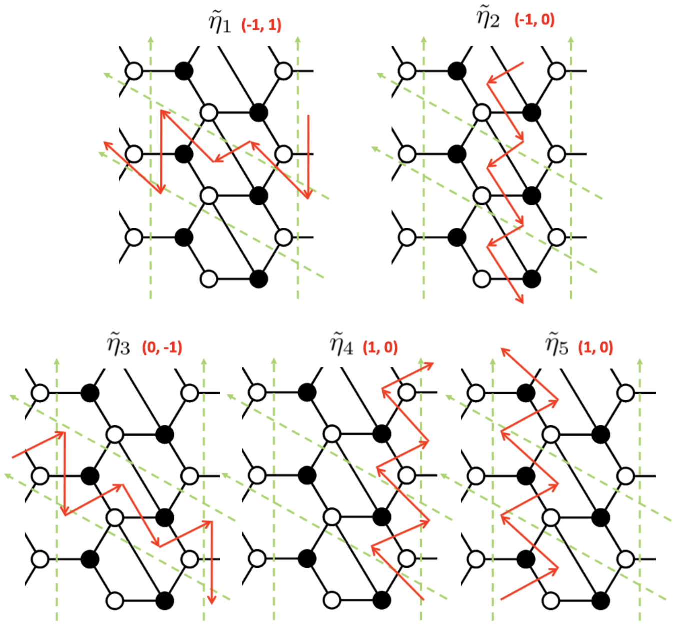

Figure 4.27. *Examples of zig-zag paths; these are the five zig-zag paths of SPP model. The y-cycle is the vertical dotted line and its positive direction is upward; the x-cycle is the transverse dotted line and its positive direction is to the left. Winding numbers of the zig-zag path are shown at the top right. Figure taken from [85].*

The fast forward algorithm

Using perfect matching technique we are able to find the points and their multiplicity of the toric diagram starting from dimer. However, is not so simple to find all perfect matchings, especially if the number of faces and edges grow. The problem of counting the number of perfect matchings is solved for any planar graph by Kasteleyn; he proposed to use the technique of the so-called Kasteleyn matrix. To define this technology we have to enumerate all the white and black vertices. Hence the Kasteleyn matrix is defined as

$$K_{bw}(x, y) = \sum_I \sigma_I A_I x^p y^q, \tag{4.51}$$

where the sum over I is the sum on all edges of a bipartite graph that connect a white vertex w to a black vertex b; p and q are the winding numbers of the edges with respect to, respectively, the x-cycle and the y-cycle of the torus; σ_I is an arbitrary sign assigned to each edge with the constrain that their product around a face is $(-1)^{m+1}$ where $2m$ is the number of edges of the face which has as edge I; moreover, A_I is an adjacency matrix defined as

$$A_I = \begin{cases} 1 & if\ the\ vertices\ b\ and\ w\ are\ connected\ by\ I \\ 0 & otherwise \end{cases}. \tag{4.52}$$

The determinant of Kasteleyn matrix, $CP(x,y) = det(K)$, is called characteristic polynomial[17] of the bipartite graph and from it we can calculate the number of perfect matchings as

$$\# \; of \; perfect \; matchings = \frac{1}{2}[-CP(1,1) + CP(1,-1) + CP(-1,1) + CP(-1,-1)]. \tag{4.53}$$

In addition to the perfect matchings number, the characteristic polynomial also gives us the toric diaggram: this is the Newton polygon of the characteristic polynomial

$$\Delta(CP(x,y)) = Convex \; hull \; of \; \{(k,l) \in \mathbb{Z}^2 | c_{k,l} \neq 0\}, \tag{4.54}$$

where $c_{k,l}$ is the numerical coefficient in front of the term $x^k y^l$. Furthermore, the value of the coefficient $c_{k,l}$ gives us the multiplicity of the toric diagram's point (k,l).

R-charges and isoradial embedding

We have a field theory specified by the quiver diagram and the superpotential that is expected to flow, due to RG flow, to a superconformal fixed point in the IR where the theory has strong coupling. An important piece of information about the superconformal field theory are the field's R-charges; hence we want to understand if exist a pictorial representation of R-charges of superconformal $U(1)_R$ at dimer level. Let us assign an R–charge R_i to each bifundamental field in the brane tiling. At the IR superconormal fixed point; each term in the superpotential satisfies the constrain

$$\sum_{i \,\in\, edge \; around \; a \; vertex} R_i = 2, \tag{4.55}$$

and this must hold for each vertex. Condition 4.55 is due to the fact that, as we know, each superpotential term must have R-charge two. The second constrain derives from the NSVZ formula: since the theory is conformal, the NSVZ beta function must vanishes and, together with the superconformal algebra relation B.8, $\Delta_i = \frac{3}{2} R_i$; this leads to the condition

$$\sum_{i \,\in\, edge \; around \; a \; face} (1 - R_i) = 2 \tag{4.56}$$

and this must hold for each face.
These constrains has a nice and interesting graphical and geometrical interpretation [109]. We multiply both equations 4.55 and 4.56 by π getting

$$\begin{aligned} \sum_{i \,\in\, edge \; around \; a \; vertex} \pi R_i &= 2\pi; \\ \sum_{i \,\in\, edge \; around \; a \; face} (\pi R_i) &= -2\pi + \pi(\# \; edges \; around \; a \; face); \end{aligned} \tag{4.57}$$

if we interpret πR_i as an angle, the first one tells us that the sum of angles around a vertex is 2π whereas the second one tells us that the sum of the internal angles in

[17]We know that the choice of x-cycle and the y-cycle of the torus is conventional, however it is possible to show that for different cycles choices the characteristic polynomial is equal up to overall multiplication by x and y.

a polygon is $\pi(\#\ edges\ around\ a\ face - 2)$; how it should be. The question now is: where are these angles in the dimer? The answer needs the notion of isoradial embedding. This is an embedding of the dimer into the plane, where the vertices of each face are on a circle of unit radius. In the following figure is represented a generic dimer (left) and its isoradial embedding (right)

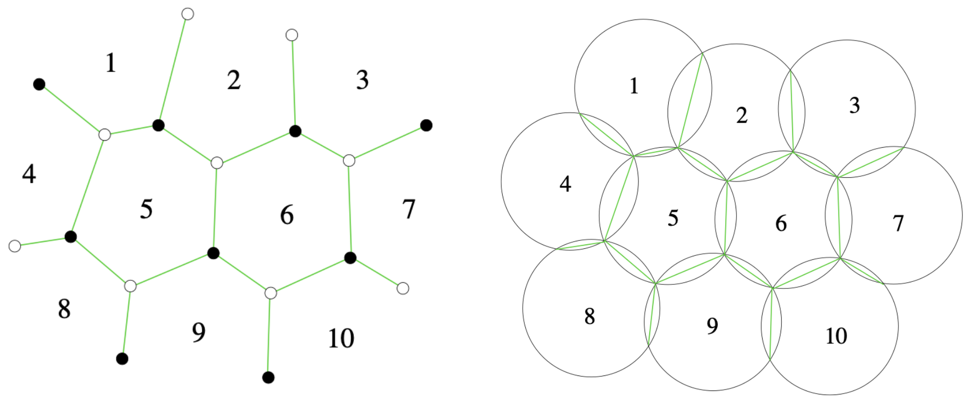

Figure 4.28. *Example of isoradial embedding; the dimer on the left is isoradial embedded in right figure. Figure taken from [109].*

Once we have the dimer isoradially embedded, we can draw the so-colled rhombus lattice which is obtained by connecting the center of the circles with the vertices of the face in the brane tiling. The rhombi in this lattice have edges of unit length since the radii of the circles are all equal to one. The rhombus lattice associated to the isoradial embedding of Figure 4.28 is given in Figure 4.29. It easy to note that rhombi are in one to one correspondence with the edges of the dimer and so with the fields of the superconformal quiver gauge theory.

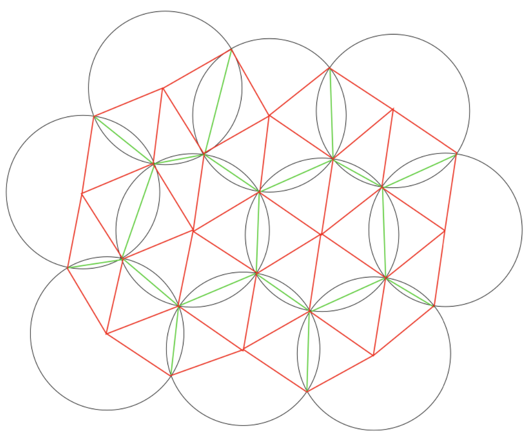

Figure 4.29. *Rhombus lattice built from isoradial embedding of Figure 4.28. The rhombi are in one to one correspondence with dimer's egdes and so with fields of the field theory. Figure taken from [109]*

If we focus on a single rhombus, as shown in the following figure, we can identify the R-charges of the fields with the angles of the rhombi $\theta_i = \pi R_i$.

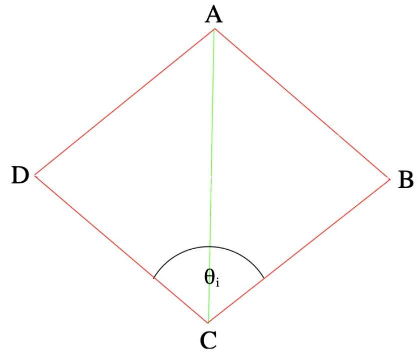

Figure 4.30. *Rhombus of the rhombi lattice, the green line is the edge of the dimer and we can associate the angle θ_i to the R-charge of the field X_i corresponding to the green edge. Figure taken from [109].*

Thanks to identification $\theta_i = \pi R_i$ and since we are on a topological torus, conditions 4.57 are certainly true; however, it is not a priori clear that an arbitrary brane tiling graph can be isoradially embedded into the plane. We can think that, if the exact R–charges are strictly greater than zero and less than one, then it is possible a good embedding of the rhombus lattice and so an isoradial embeddable dimer. If not, the corresponding rhombus becomes degenerate.

Higgsing and unhiggsing

The Higgs mechanism has a natural interpretation in the context of brane tiling picture as shown in [110]: giving a non zero vacuum expectation value to a gauge field in brane tiling A and integrating out resulting quadratic mass terms in the superpotential we obtains a new brane tiling B whose moduli space is a different toric CY threefold from the one of brane tiling A. Since massless fields correspond to strings between branes in the same position and we know that a massive field is possible only if one brane moves away from the others, integrating out massive field correspond to eliminate the associated edge from the dimer. This results in an effective merger between two adjacent faces analogous of combining two gauge groups into one, and corresponding to the break of the gauge group due to the moving away of the brane.
Let us give the example of $\frac{\mathbb{C}^3}{\mathbb{Z}_2\times\mathbb{Z}_2}$ whit orbifold action $((0,1,1),(1,0,1))$; this theory when higgsed gives SPP, in the next page the graphical interpretation.
The superpotential for $\frac{\mathbb{C}^3}{\mathbb{Z}_2\times\mathbb{Z}_2}$ model model is

$$W_A = Tr\big(X_{42}X_{23}X_{34} + X_{31}X_{14}X_{43} + X_{24}X_{41}X_{12} + X_{13}X_{32}X_{21} + \\ - X_{42}X_{21}X_{14} - X_{31}X_{12}X_{23} - X_{24}X_{43}X_{32} - X_{13}X_{34}X_{41}\big). \tag{4.58}$$

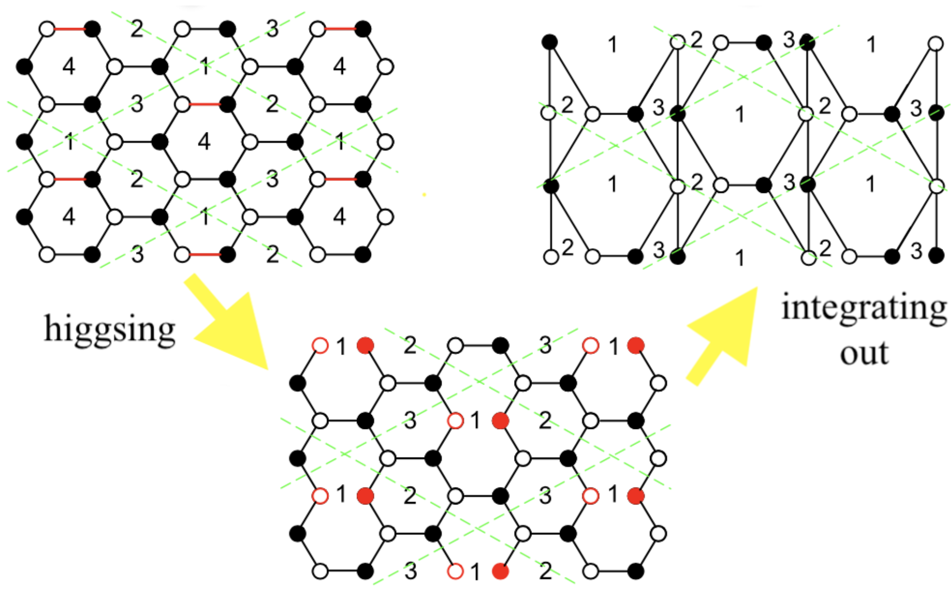

Figure 4.31. *Higgsing and integrating out for $\frac{\mathbb{C}^3}{\mathbb{Z}_2\times\mathbb{Z}_2}$ model. We start with dimer of $\frac{\mathbb{C}^3}{\mathbb{Z}_2\times\mathbb{Z}_2}$, then eliminating the edge corresponding to X_{14} we obtain a merge of faces 1 and 4.This is shown in the second dimer where mass term of superpotential are represented with red vertex. Integrating out massive fields we obtain the third dimer which correspond with SPP model's dimer. Figure taken and modified from [85].*

Giving a vacuum expectation value $\langle X_{14}\rangle$=1 to field X_{14} the superpotential becomes

$$\begin{aligned} W_A = Tr(&X_{12}X_{23}X_{31} + X_{31}X_{13} + X_{21}X_{11}X_{12} + X_{13}X_{32}X_{21}+ \\ &- X_{12}X_{21} - X_{31}X_{12}X_{23} - X_{21}X_{13}X_{32} - X_{13}X_{31}X_{11}) \end{aligned} \tag{4.59}$$

and we can integrate out quadratic terms thanks to equations $X_{13} = X_{12}X_{23}$ and $X_{12} = X_{13}X_{32}$ getting the new superpotential

$$W_B = Tr(X_{13}X_{32}X_{23}X_{31} + X_{21}X_{11}X_{12} - X_{21}X_{12}X_{23}X_{32} - X_{13}X_{31}X_{11}) \tag{4.60}$$

It is clear that we can do this steps backwards, this procedure is named unhiggsing. Moreover, there is an interesting relation between higgsing, unhiggsing and blow up, blow down: the blow up of a toric diagram's point is translated to an unhiggsing of the field theory on the D-branes; in the same way blow down corresponds to the higgsing. Hence, for example, since the dP surfaces are obtained by blow up of $\mathbb{CP}^2$, this means that the field theories dual to the cone over del Pezzo surfaces should be obtained by subsequent unhiggsing.

Toric duality

Toric duality is nothing but Seiberg duality[18]: two SUSY theories are called toric dual if in the UV they have different lagrangians with a different field content and superpotential but flow to the same IR fixed point in the conformal window. The

[18] For a refresh on Seiberg duality see 1.3.3.

moduli space of the two theories is identical and so the toric diagram is the same[19]; however multiplicities of internal toric points and hence GLSM fields with zero R-charge can differ. These dual theories are called toric phases. Let us show how this works with the example of $\mathbb{F}_0$; below are reported the dimer and the quiver of the theory.

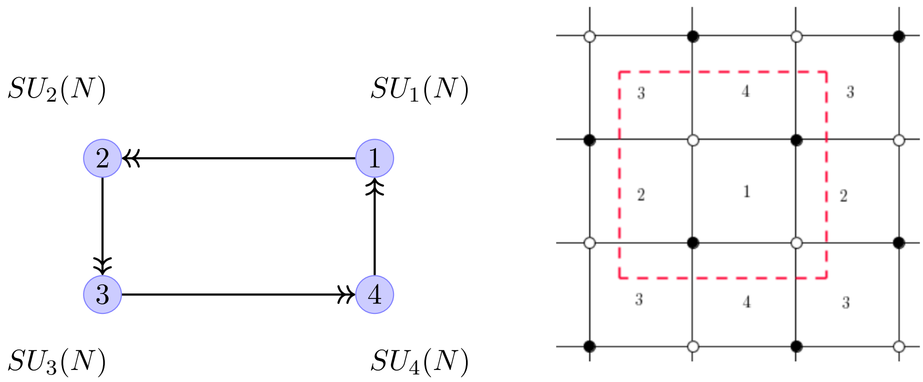

Figure 4.32. *Quiver and dimer for the cone over* $\mathbb{F}_0$*.*

The superpotential is given by

$$W_e = Tr\big(X^1_{41}X^2_{12}X^1_{23}X^2_{34} - X^1_{41}X^1_{12}X^1_{23}X^1_{34} + X^1_{12}X^2_{23}X^1_{34}X^2_{41} - X^2_{41}X^2_{12}X^2_{23}X^2_{34}\big). \tag{4.61}$$

Let us Seiberg dualize gauge group $SU_2(N)$, the dual theory will have a gauge group $SU_2(N_f - N)$ but since $N_f = 2N$ we will have again $SU_2(N)$; moreover, it is easy to check that we are in the conformal window $\frac{3}{2}N < 2N < 3N$. Recall that in the magnetic theory we have also the mesons as independent degrees of freedom and these are given by $M_{ij} = X^i_{12}X^j_{23}$ in the electric theory. Fields in the electric theory that transform not in a trivial way under $SU_2(N)$ are replaced by new fields $q^1_{12}, q^2_{12}, \bar{q}^1_{23}, \bar{q}^2_{23}$ and the superpotential of the magnetic theory is written from the one of the electric theory in terms of mesons, old and new fields

$$W_m = Tr\big(M_{21}X^2_{34}X^1_{41} - M_{11}X^1_{34}X^1_{41} + M_{12}X^1_{34}X^2_{41} - M_{22}X^2_{34}X^2_{41} + M_{ij}q^i_{12}\bar{q}^j_{23}\big). \tag{4.62}$$

This duality can be interpreted graphically as shown in the Figure 4.33 where we pass to four tetravalent vertices to eight trivalent ones.

We can observe that under toric duality, the number of gauge groups remains constant while the number of bifundamental fields and superpotential terms increase both by four. The magnetic theory quiver diagram have to contain also the mesons and it is draw below in Figure 4.34.

[19]It is the same up to $SL(2, \mathbb{Z})$ transformations.

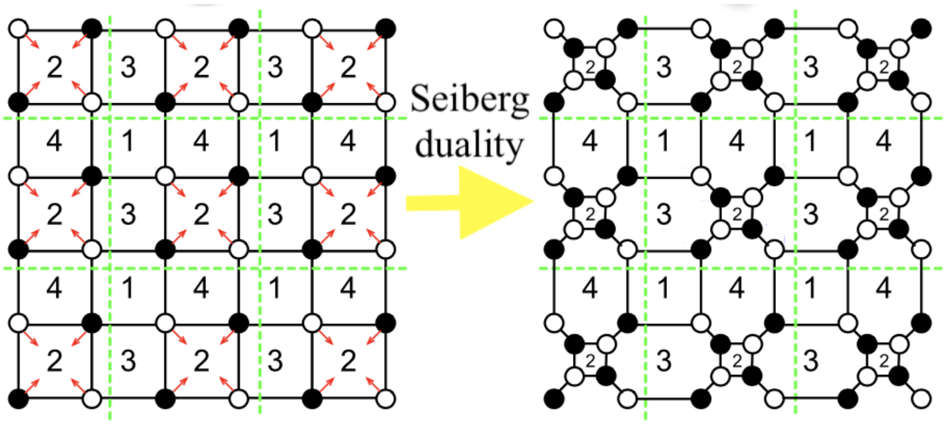

Figure 4.33. *Graphical representation of toric (Seiberg) duality. These are the toric phases of $\mathbb{F}_0$ cone model. Figure taken and modified from [85].*

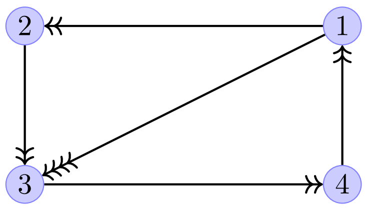

Figure 4.34. *Quiver diagram for the magnetic dual theory of $\mathbb{F}_0$ cone model.*

As we said before, under toric duality toric diagram does not change shape but only its multiplicity[20], this fact can be verified by explicit calculation of Kasteleyn matrix and we report the toric diagram for the two phases.

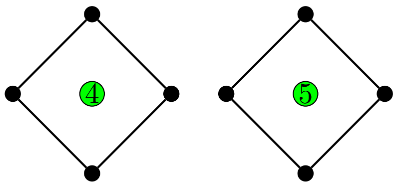

Figure 4.35. *Toric diagrams for the two toric phases of $\mathbb{F}_0$ cone model. On the left we have the electric theory while on the right the magnetic one.*

A general constrain on toric diality is that Seiberg duality maps a toric theory into a toric theory only when it acts on a gauge group with two incoming and two outgoing arrows. Seiberg duality operation on a gauge group G can be represented on the bipartite graph; adding the four fundamental mesons and coupling them with the

[20] Recall that a theory is inconsistent if the vertex points have multiplicity grater than one.

magnetic quarks corresponds to the operation in Figure 4.36. The important point is that this is a local operation: the move on the bipartite graph only affects the neighborhood of face G; the rest of the diagram does not know anything about this operation.

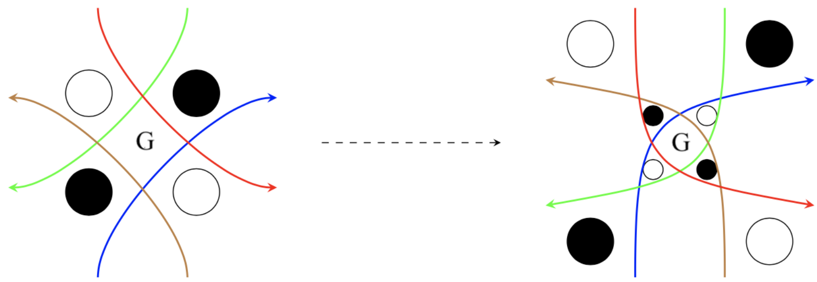

Figure 4.36. *Bipartite graph's move corresponding to Seiberg dialize gauge gruop G. We have four new superpotential terms. This move take fixed the winding numbers and preserve bipartness of the graph.*

Obviously, nothing forbids us to Seiberg dualize a gauge group with more or less than two incoming and outgiong arrows but the magnetic theory will not toric anymore. The information about the web of Seiberg dualities on a given quiver gauge theory con be encoded in its duality tree diagram [111]; each node represents a phase of the theory and two nodes are connected if they are obtained one from the other through exactly one Seiberg duality. The example for the quiver theory dual to a D3-branes configuration with geometry given by the cone over $\mathbb{F}_0$ is reported below.

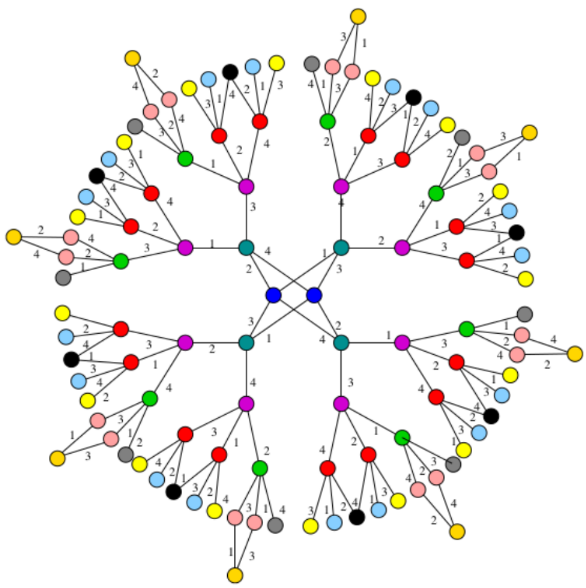

Figure 4.37. *Duality tree diagram for cone over $\mathbb{F}_0$ model, each node is a toric phase of the theory and they are connected by one line indicating that is possible pass to one phase to another with a single Seiberg duality; numbers indicate the dualize gauge group.*

4.3 The thirty reflexive polygons quiver theories

Previously we have studied reflexive polygons, now the natural question to ask is: which supersymmetric quiver gauge theories exist whose space of vacua correspond to the sixteen reflexive polygons? To answer to this question we need higgsing, unhiggsing and toric duality tools in order to understand how many and which one they are. In the end turns out that there are thirty SUSY quiver theories with reflexive associate toric diagram; these are completely classified by Seong in [85] and we report here only the toric diagrams with multiplicity of the model and its possible toric phases, their dimers (taken from [85]) and quiver diagrams. We will refer to Figure 4.13 to enumerate the models.

Model 1: $\frac{\mathbb{C}^3}{\mathbb{Z}_3\times\mathbb{Z}_3}$ **with orbifold action** $((1,0,2),(0,1,2))$

We have only one toric phase with twentyseven fields, and the superpotential contains eighteen terms:

$$\begin{aligned} W = Tr\big(&+ X_{15}X_{56}X_{61} + X_{29}X_{91}X_{12} + X_{31}X_{18}X_{83} + X_{42}X_{23}X_{34}+ \\ &+ X_{53}X_{37}X_{75} + X_{67}X_{72}X_{26} + X_{78}X_{89}X_{97} + X_{86}X_{64}X_{48}+ \\ &+ X_{94}X_{45}X_{59} - X_{15}X_{59}X_{91} - X_{29}X_{97}X_{72} - X_{31}X_{12}X_{23}+ \\ &- X_{42}X_{26}X_{64} - X_{53}X_{34}X_{45} - X_{67}X_{75}X_{56} - X_{78}X_{83}X_{37}+ \\ &- X_{86}X_{61}X_{18} - X_{94}X_{48}X_{89}\big) \end{aligned} \tag{4.63}$$

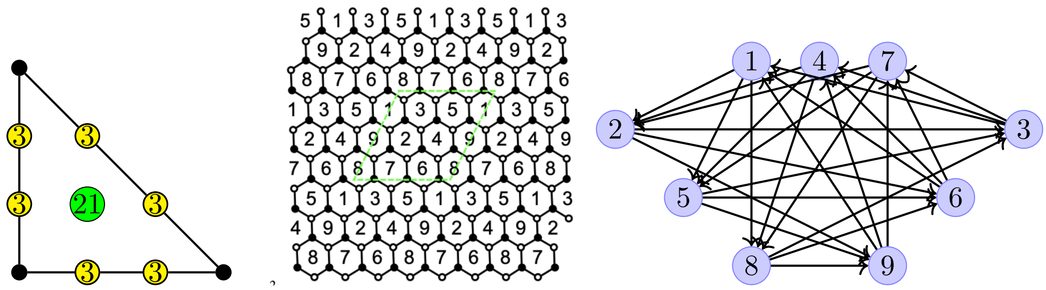

Figure 4.38. *Toric diagram, dimer and quiver for the only one toric phase of model 1.*

Model 2: $\frac{\mathbb{C}^3}{\mathbb{Z}_4\times\mathbb{Z}_2}$ **with orbifold action** $((1,0,3),(0,1,1))$

We have only one toric phase with twentyfour fields and the sixteen terms superpotential is

$$\begin{aligned} W = Tr\big(&+ X_{17}X_{72}X_{21} + X_{28}X_{81}X_{12} + X_{31}X_{14}X_{43} + X_{42}X_{23}X_{34}+ \\ &+ X_{53}X_{36}X_{65} + X_{64}X_{54}X_{56} + X_{75}X_{58}X_{87} + X_{86}X_{67}X_{78}+ \\ &- X_{17}X_{78}X_{81} - X_{28}X_{87}X_{72} - X_{31}X_{12}X_{23} - X_{42}X_{21}X_{14}+ \\ &- X_{53}X_{34}X_{45} - X_{64}X_{43}X_{36} - X_{75}X_{56}X_{67} - X_{86}X_{65}X_{58}\big) \end{aligned} \tag{4.64}$$

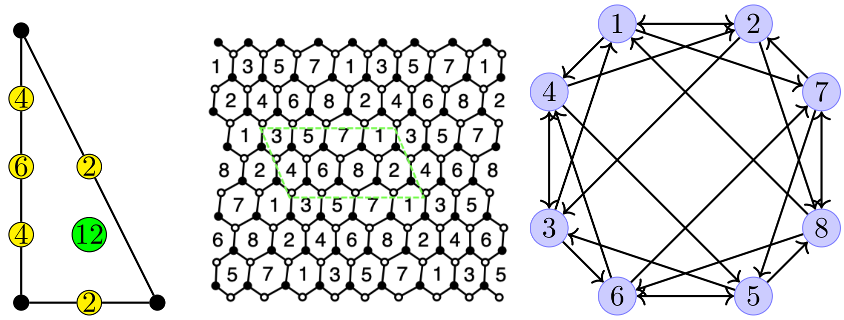

Figure 4.39. *Toric diagram, dimer and quiver for the only one toric phase of model 2.*

Model 3: cone over $\frac{L_{1,3,1}}{\mathbb{Z}_2}$ with orbifold action $(0,1,1,1)$

We have two toric phases, the superpotentials are given by

$$\begin{aligned} W_a = Tr\big(&+ X_{31}X_{18}X_{83} + X_{32}X_{27}X_{73} + X_{53}X_{37}X_{75} + X_{78}X_{81}X_{17}+ \\ &+ X_{14}X_{45}X_{56}X_{61} + X_{62}X_{24}X_{48}X_{86}+ \\ &- X_{32}X_{24}X_{45}X_{53} - X_{62}X_{27}X_{75}X_{56}+ \\ &- X_{14}X_{48}X_{81} - X_{31}X_{17}X_{73} - X_{78}X_{83}X_{37} - X_{86}X_{61}X_{18}\big) \end{aligned} \tag{4.65}$$

and

$$\begin{aligned} W_b = Tr\big(&+ X_{31}X_{18}X_{83} + X_{42}X_{23}X_{34} + X_{53}X_{37}X_{75} + X_{67}X_{72}X_{26}+ \\ &- X_{14}X_{48}X_{81} - X_{42}X_{26}X_{64} - X_{53}X_{34}X_{45} - X_{67}X_{75}X_{56}+ \\ &+ X_{78}X_{81}X_{17} + X_{86}X_{64}X_{48} - X_{78}X_{83}X_{37} - X_{86}X_{61}X_{18}+ \\ &+ X_{14}X_{45}X_{56}X_{61} - X_{17}X_{72}X_{23}X_{31}\big). \end{aligned} \tag{4.66}$$

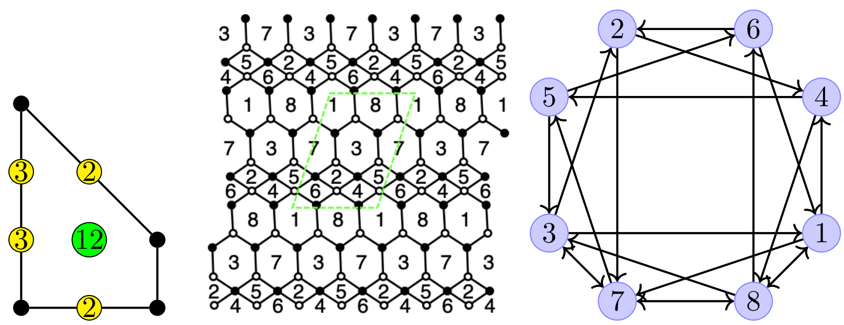

Figure 4.40. *Toric diagram, dimer and quiver for phase a of model 3.*

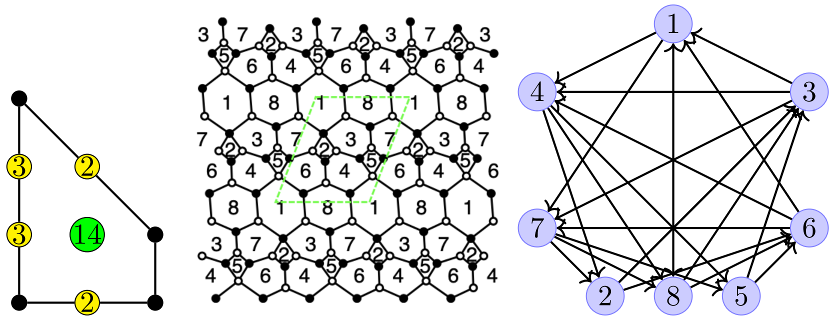

Figure 4.41. *Toric diagram, dimer and quiver for phase b of model 3.*

Model 4: cone over $\mathbf{PdP}_5$

We have four toric phases, the superpotentials are:

$$\begin{aligned} W_a = Tr\big(&+ X_{23}X_{38}X_{81}X_{12} + X_{41}X_{16}X_{63}X_{34} + X_{67}X_{74}X_{45}X_{56} + X_{85}X_{52}X_{27}X_{78} + \\ &- X_{27}X_{74}X_{41}X_{12} - X_{45}X_{52}X_{23}X_{34} - X_{63}X_{38}X_{85}X_{56} - X_{81}X_{16}X_{67}X_{78}\big); \end{aligned} \quad (4.67)$$

$$\begin{aligned} W_b = Tr\big(&+ X_{23}X_{38}X_{82} + X_{45}X_{56}X_{64} + X_{63}X_{34}X_{46} + X_{85}X_{52}X_{28} + \\ &+ X_{21}X_{14}X_{47}X_{72} + X_{61}X_{18}X_{87}X_{76} + \\ &- X_{45}X_{52}X_{23}X_{34} - X_{63}X_{38}X_{85}X_{56} + \\ &- X_{21}X_{18}X_{82} - X_{47}X_{76}X_{64} - X_{87}X_{72}X_{28} - X_{61}X_{14}X_{46}\big); \end{aligned} \quad (4.68)$$

$$\begin{aligned} W_c = Tr\big(&+ X_{21}X_{14}X_{42} + X_{23}X_{38}X_{82} + X_{61}X_{18}X_{86} + X_{63}X_{34}X_{46} + \\ &+ X_{67}X_{74}X_{45}X_{56} + X_{85}X_{52}X_{27}X_{78} + \\ &- X_{45}X_{52}X_{23}X_{34} - X_{63}X_{38}X_{85}X_{56} + \\ &- X_{21}X_{18}X_{82} - X_{27}X_{74}X_{42} - X_{61}X_{14}X_{46} - X_{67}X_{78}X_{86}\big); \end{aligned} \quad (4.69)$$

$$\begin{aligned} W_d = Tr\big(&+ X_{21}X_{14}X^1_{42} + X_{23}X_{38}X^1_{82} + X_{25}X_{54}X^2_{42} + X_{27}X_{78}X^2_{82} + \\ &+ X_{61}X_{18}X^1_{86} + X_{63}X_{34}X^1_{46} + X_{65}X_{58}X^2_{86} + X_{67}X_{74}X^2_{46} + \\ &- X_{21}X_{18}X^1_{82} - X_{23}X_{34}X^2_{42} - X_{25}X_{58}X^2_{82} - X_{27}X_{74}X^1_{42} + \\ &- X_{61}X_{14}X^1_{46} - X_{63}X_{38}X^2_{86} - X_{65}X_{54}X^2_{46} - X_{67}X_{78}X^1_{86}\big). \end{aligned} \quad (4.70)$$

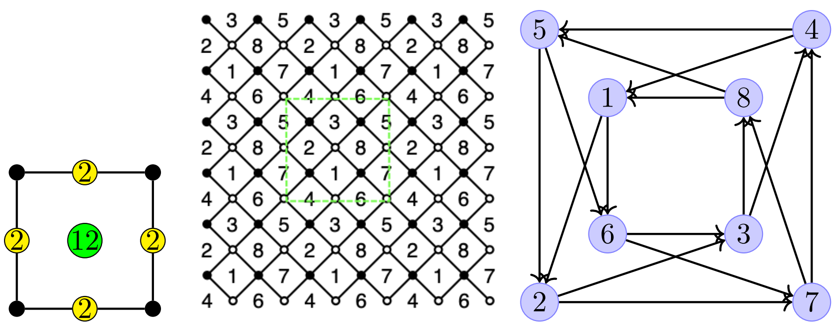

Figure 4.42. *Toric diagram, dimer and quiver for phase a of model 4.*

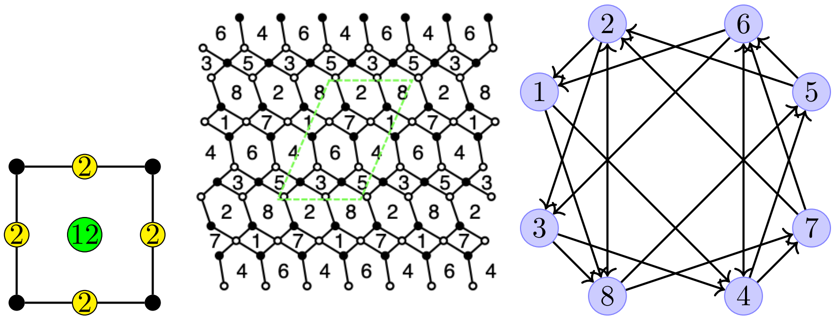

Figure 4.43. *Toric diagram, dimer and quiver for phase b of model 4.*

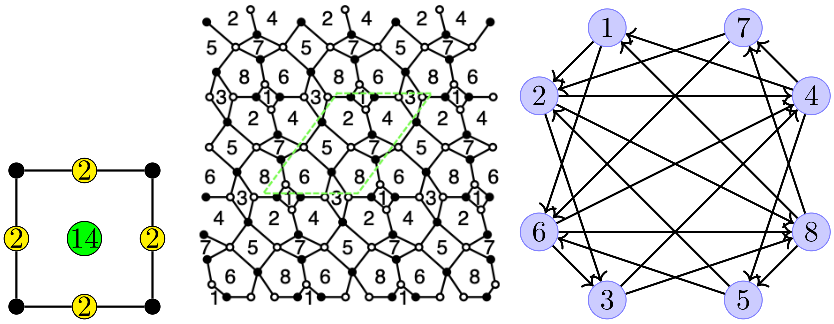

Figure 4.44. *Toric diagram, dimer and quiver for phase c of model 4.*

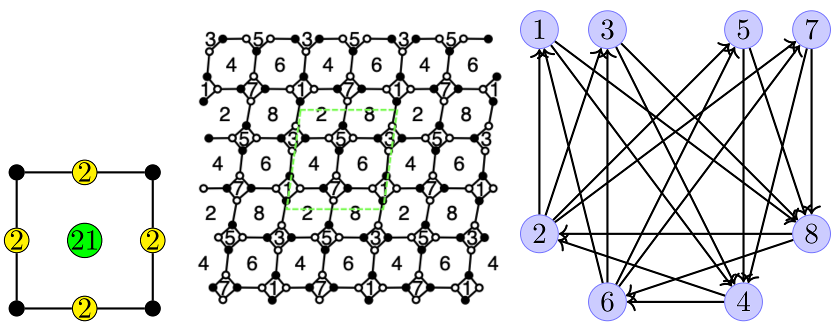

Figure 4.45. *Toric diagram, dimer and quiver for phase d of model 4.*

Model 5: cone over $\mathbf{PdP}_{4b}$

This model has only one phase with nineteen fields, the superpotential contains twelve terms and it is given by

$$\begin{aligned} W = Tr\big(&+ X_{21}X_{17}X_{72} + X_{42}X_{26}X_{64} + X_{56}X_{62}X_{25} + X_{67}X_{71}X_{16} + X_{75}X_{53}X_{37} + \\ &- X_{13}X_{37}X_{71} - X_{16}X_{62}X_{21} - X_{56}X_{64}X_{45} - X_{67}X_{72}X_{26} - X_{75}X_{51}X_{17} + \\ &+ X_{13}X_{34}X_{45}X_{51} - X_{25}X_{53}X_{34}X_{42}\big). \end{aligned} \tag{4.71}$$

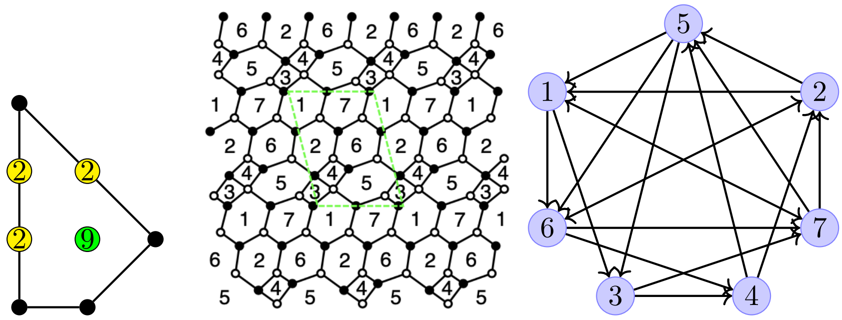

Figure 4.46. *Toric diagram, dimer and quiver for the only toric phase of model 5.*

Model 6: cone over $\mathbf{PdP}_{4a}$

Model 6 has three toric phases and the superpotentials are

$$\begin{aligned} W_a = Tr\big(&+ X_{32}X_{27}X_{73} + X_{14}X_{45}X_{56}X_{61} + X_{31}X_{17}X_{75}X_{53} + X_{62}X_{24}X_{47}X_{76} + \\ &- X_{76}X_{61}X_{17} - X_{31}X_{14}X_{47}X_{73} - X_{32}X_{24}X_{45}X_{53} - X_{62}X_{27}X_{75}X_{56}\big); \end{aligned} \tag{4.72}$$

$$\begin{aligned} W_b = Tr\big(&+ X_{42}X_{23}X_{34} + X_{67}X_{72}X_{26} + X_{76}X_{64}X_{47} + \\ &+ X_{14}X_{45}X_{56}X_{61} + X_{31}X_{17}X_{75}X_{53} + \\ &- X_{67}X_{75}X_{56} - X_{76}X_{61}X_{17} - X_{42}X_{26}X_{64} - X_{53}X_{34}X_{45} + \\ &- X_{14}X_{47}X_{72}X_{23}X_{31}\big); \end{aligned} \tag{4.73}$$

$$\begin{aligned} W_c = Tr\big(&+ X_{41}X_{13}X^2_{34} + X_{45}X_{23}X^1_{34} + X_{45}X_{56}X^2_{64} + X_{67}X_{72}X_{26} + X_{75}X_{53}X_{37} + \\ &- X_{41}X_{16}X^2_{64} - X_{42}X_{26}X^1_{64} - X_{45}X_{53}X^1_{34} - X_{67}X_{75}X_{56} - X_{71}X_{13}X_{37} + \\ &+ X_{47}X_{71}X_{16}X^1_{64} - X_{47}X_{72}X_{23}X^2_{34}\big). \end{aligned} \tag{4.74}$$

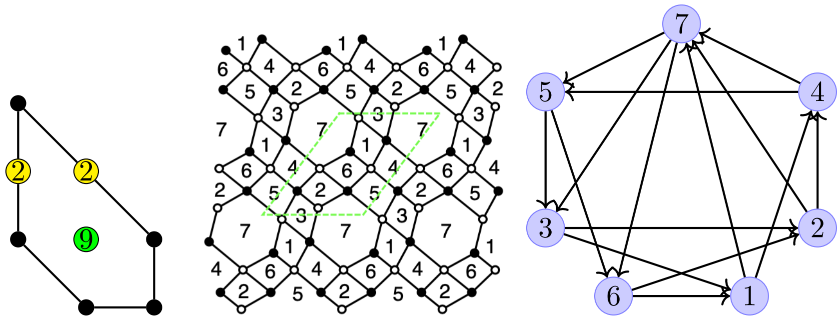

Figure 4.47. *Toric diagram, dimer and quiver for phase a of model 6.*

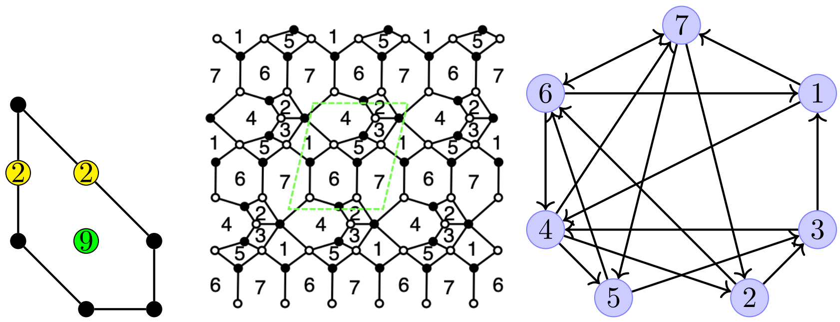

Figure 4.48. *Toric diagram, dimer and quiver for phase b of model 6.*

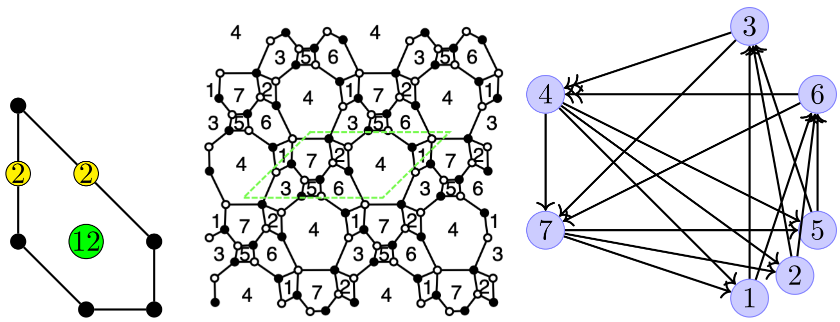

Figure 4.49. *Toric diagram, dimer and quiver for phase c of model 6.*

Model 7: cone over $\mathbf{PdP}_{3a}$

We have only one toric phase with eighteen fields, the superpotential has twelve terms and it is given by

$$\begin{aligned} W = Tr\big(&+ X_{12}X_{26}X_{61} + X_{63}X_{34}X_{46} + X_{24}X_{43}X_{32} + \\ &+ X_{35}X_{51}X_{13} + X_{41}X_{15}X_{54} + X_{56}X_{62}X_{25} + \\ &- X_{12}X_{25}X_{51} - X_{63}X_{32}X_{26} - X_{24}X_{46}X_{62} + \\ &- X_{35}X_{54}X_{43} - X_{41}X_{13}X_{34} - X_{56}X_{61}X_{15}\big). \end{aligned} \tag{4.75}$$

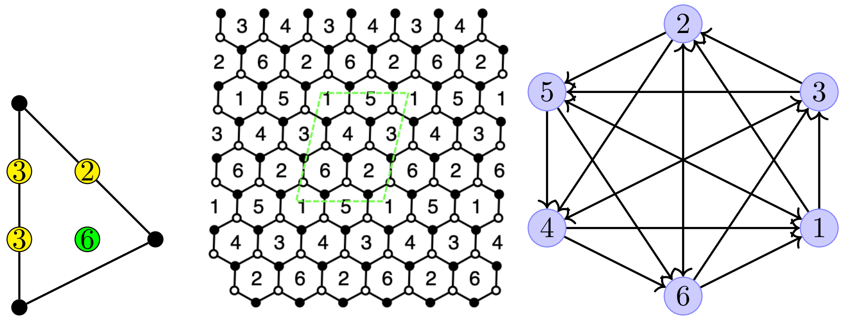

Figure 4.50. *Toric diagram, dimer and quiver for the only toric phase of model 7.*

Model 8: cone over $\mathbf{PdP}_{3c}$

Model 8 has two toric phases and superpotentials are

$$W_a = Tr\big(+ X_{56}X_{62}X_{25} + X_{65}X_{53}X_{36} + X_{13}X_{34}X_{45}X_{51} + X_{21}X_{16}X_{64}X_{42} + \\ - X_{56}X_{64}X_{45} - X_{65}X_{51}X_{16} - X_{13}X_{36}X_{62}X_{21} - X_{25}X_{53}X_{34}X_{42}\big); \tag{4.76}$$

$$W_b = Tr\big(+ X_{31}X_{12}X_{23} + X_{56}X_{62}X_{25} + X_{64}X_{42}X_{26} + X_{34}X_{45}X^2_{53} + \\ - X_{31}X_{15}X^2_{53} - X_{36}X_{62}X_{23} - X_{56}X_{64}X_{45} - X_{61}X_{12}X_{26} + \\ + X_{61}X_{15}X^1_{53}X_{36} - X_{25}X^1_{53}X_{34}X_{42}\big). \tag{4.77}$$

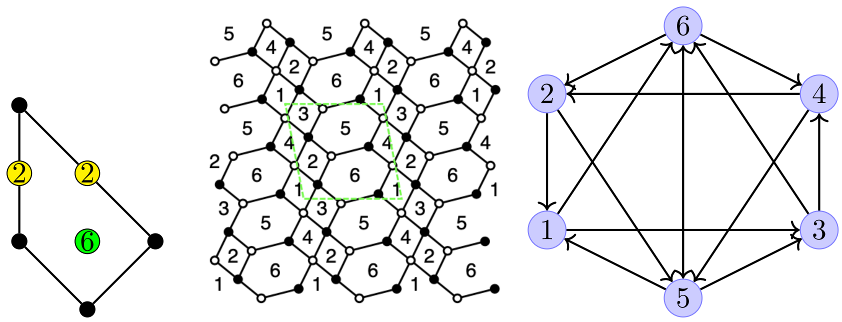

Figure 4.51. *Toric diagram, dimer and quiver for phase a of model 8.*

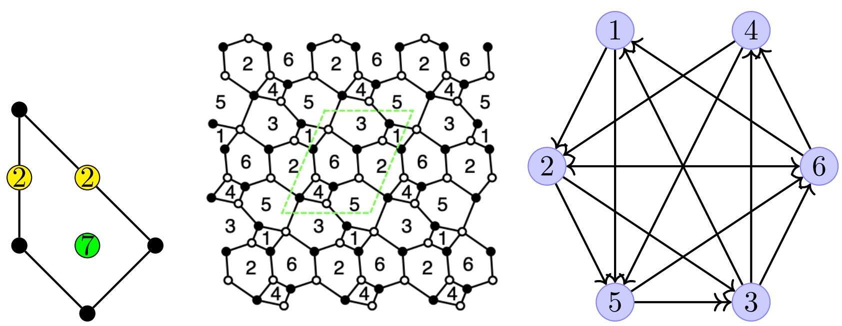

Figure 4.52. *Toric diagram, dimer and quiver for phase b of model 8.*

Model 9: cone over PdP$_{3b}$

This model has three toric phases with superpotentials

$$\begin{aligned} W_a = Tr\big(&+ X_{12}X_{26}X_{61} + X_{25}X_{53}X_{32} + X_{42}X_{21}X_{14} + X_{13}X_{34}X_{46}X_{65}X_{51} + \\ &- X_{13}X_{32}X_{21} - X_{25}X_{51}X_{12} - X_{46}X_{61}X_{14} - X_{26}X_{65}X_{53}X_{34}X_{42}\big); \end{aligned} \tag{4.78}$$

$$\begin{aligned} W_b = Tr\big(&+ X^2_{25}X_{53}X_{32} + X_{56}X_{62}X^1_{25} + X_{13}X_{34}X_{45}X_{51} + X_{21}X_{16}X_{64}X_{42} + \\ &- X_{13}X_{32}X_{21} - X_{56}X_{64}X_{45} - X_{16}X_{62}X^2_{25}X_{51} - X^1_{25}X_{53}X_{34}X_{42}\big); \end{aligned} \tag{4.79}$$

$$\begin{aligned} W_c = Tr\big(&+ X_{21}X_{16}X^2_{62} + X_{24}X_{43}X^2_{32} + X^2_{25}X_{53}X^1_{32} + X_{51}X_{13}X_{35} + \\ &- X_{13}X^1_{32}X_{21} - X_{24}X_{46}X^2_{62} - X^1_{25}X_{53}X^2_{32} - X_{54}X_{43}X_{35} + \\ &+ X_{54}X_{46}X^1_{62}X^1_{25} - X_{16}X^1_{62}X^1_{25}X_{51}\big). \end{aligned} \tag{4.80}$$

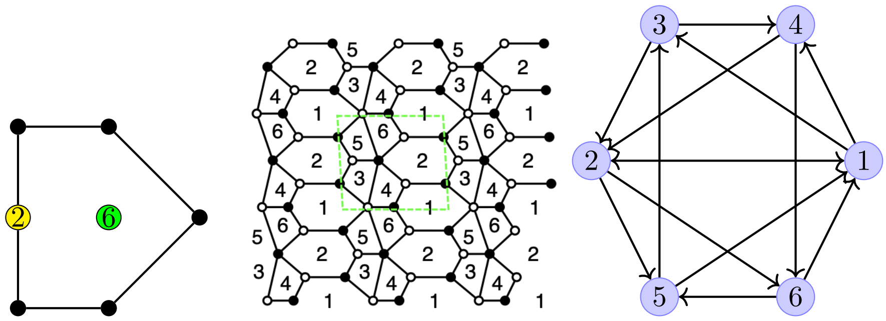

Figure 4.53. *Toric diagram, dimer and quiver for phase a of model 9.*

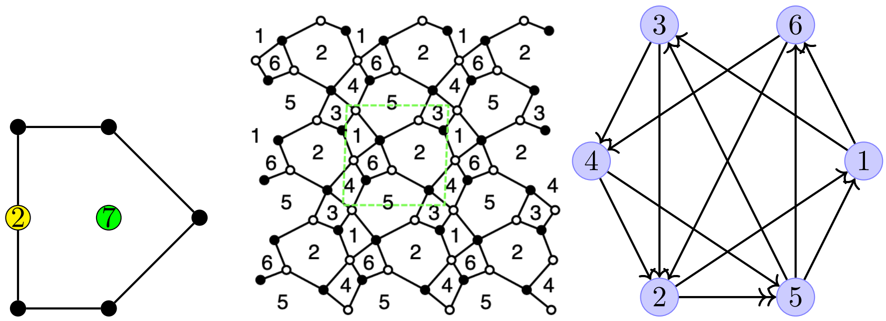

Figure 4.54. *Toric diagram, dimer and quiver for phase b of model 9.*

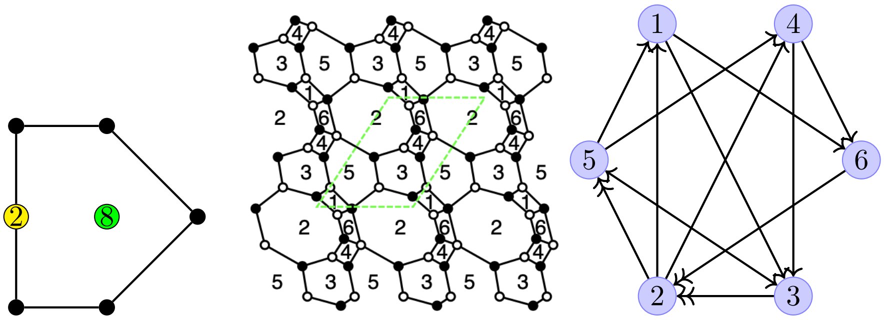

Figure 4.55. *Toric diagram, dimer and quiver for phase c of model 9.*

Model 10: cone over dP_3

In this model we have four toric phases and superpotential are given by

$$W_a = Tr\big(+ X_{13}X_{32}X_{21} + X_{56}X_{64}X_{45} + X_{43}X_{35}X_{52}X_{26}X_{61}X_{14} + \\ - X_{13}X_{35}X_{56}X_{61} - X_{14}X_{45}X_{52}X_{21} - X_{26}X_{64}X_{43}X_{32}\big); \quad (4.81)$$

$$W_b = Tr\big(+ X_{31}X_{15}X_{53} + X_{42}X_{23}X_{34} + X_{56}X_{64}X^2_{45} + X_{52}X_{26}X_{61}X_{14}X_{45^1} + \\ - X_{42}X_{26}X_{64} - X_{53}X_{34}X^1_{45} - X_{56}X_{61}X_{15} - X_{14}X^2_{45}X_{52}X_{23}X_{31}\big); \quad (4.82)$$

$$W_c = Tr\big(+ X_{41}X_{13}X^2_{34} + X_{42}X_{23}X^1_{34} + X^1_{45}X_{52}X_{26}X^2_{64} + X_{51}X_{16}X^1_{64}X^2_{45} + \\ - X_{41}X_{16}X^2_{64} - X_{42}X_{26}X^1_{64} - X^2_{45}X_{52}X_{23}X^2_{34} - X_{51}X_{13}X^1_{34}X^1_{45}\big); \quad (4.83)$$

$$W_d = Tr\big(+ X_{15}X^1_{54}X^2_{41} + X_{25}X^2_{54}X^2_{42} + X_{26}X^2_{64}X^3_{42} + X^1_{41}X_{13}X^2_{34} + \\ + X_{16}X^1_{64}X^3_{41} + X^1_{42}X_{23}X^1_{34} - X_{15}X^2_{54}X^3_{41} - X_{13}X^1_{34}X^2_{41} + \\ - X_{23}X^2_{34}X^2_{42} - X_{25}X^1_{54}X^3_{42} - X^1_{41}X_{16}X^2_{64} - X^1_{42}X_{26}X^1_{64}\big). \quad (4.84)$$

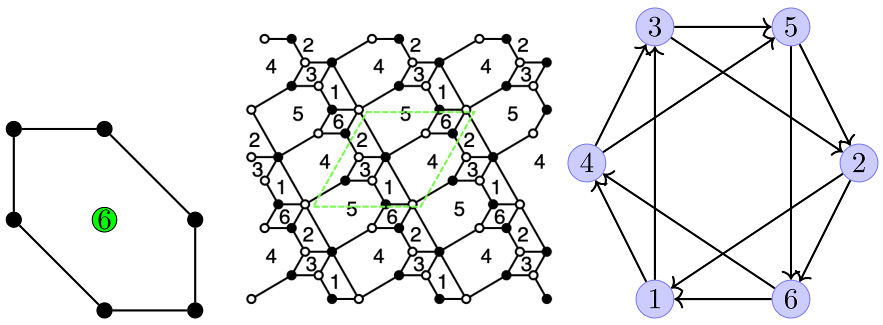

Figure 4.56. *Toric diagram, dimer and quiver for phase a of model 10.*

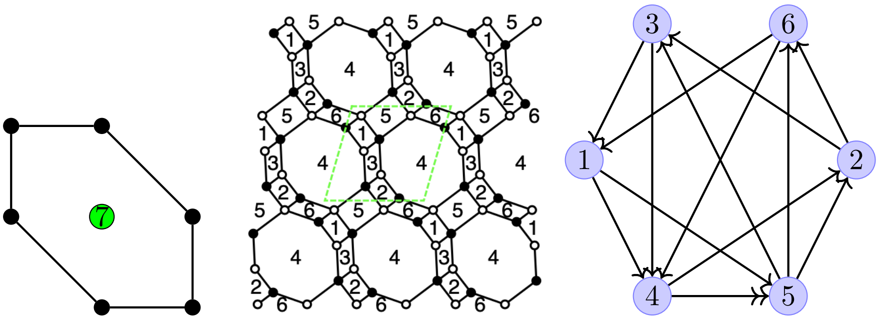

Figure 4.57. *Toric diagram, dimer and quiver for phase b of model 10.*

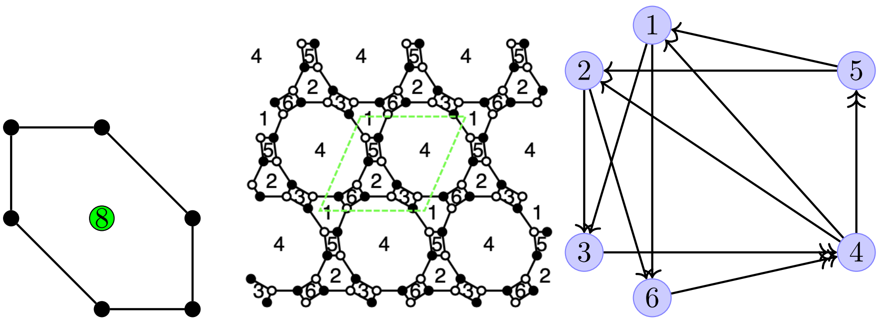

Figure 4.58. *Toric diagram, dimer and quiver for phase c of model 10.*

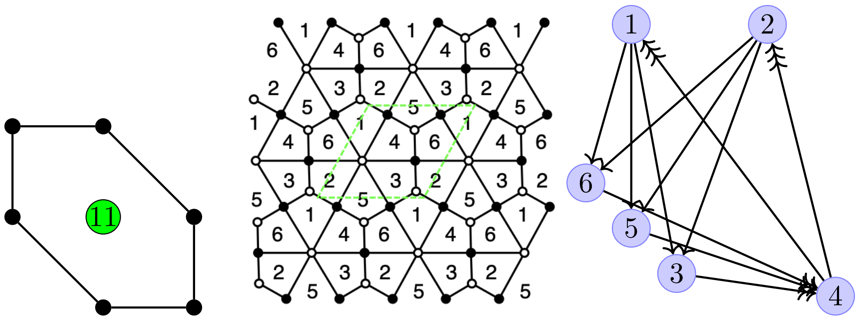

Figure 4.59. *Toric diagram, dimer and quiver for phase d of model 10.*

Model 11: cone over PdP_2

This model has only one toric phase with thirteen fields and with superpotential, which contains eight terms, given by

$$W = Tr\big(+ X_{21}X_{14}X_{42} + X_{53}X_{32}X_{25}^{2} + X_{51}^{2}X_{12}X_{25}^{1} + X_{13}X_{34}X_{45}X_{51}^{1} + - X_{13}X_{32}X_{21} - X_{14}X_{45}X_{51}^{2} - X_{51}^{1}X_{12}X_{25}^{2} - X_{53}X_{34}X_{42}X_{25}^{1}\big). \tag{4.85}$$

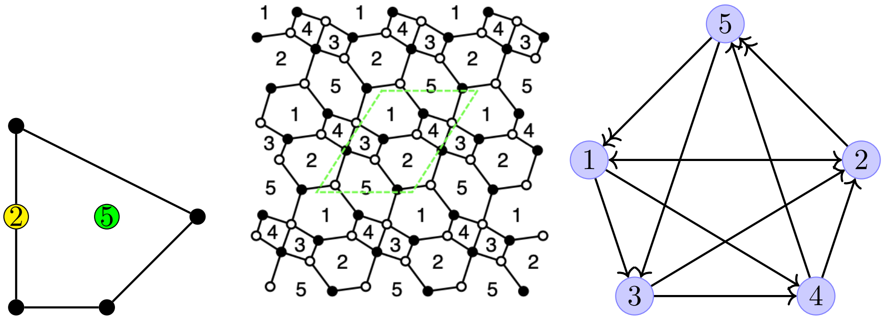

Figure 4.60. *Toric diagram, dimer and quiver for the only toric phase of model 11.*

Model 12: cone over dP_2

Model 12 has two toric phases whose superpotentials are

$$\begin{aligned} W_a = Tr\big(&+ X_{21}X_{14}X^1_{42} + X^2_{25}X_{53}X_{32} - X_{13}X_{32}X_{21} + \\ &+ X^2_{42}X^1_{25}X_{51}X_{13}X_{34} - X_{14}X^2_{42}X^2_{25}X_{51} - X^1_{25}X_{53}X_{34}X^1_{42}\big); \end{aligned} \tag{4.86}$$

$$\begin{aligned} W_b = Tr\big(&+ X_{15}X^2_{52}X^2_{21} + X^1_{21}X_{14}X^1_{42} + X_{35}X^1_{52}X_{23} + X_{13}X_{34}X^2_{42}X^3_{21} + \\ &- X_{14}X^2_{42}X^2_{21} - X_{15}X^1_{52}X^3_{21} - X_{34}X^1_{42}X_{23} - X^1_{21}X_{13}X_{35}X^2_{52}\big). \end{aligned} \tag{4.87}$$

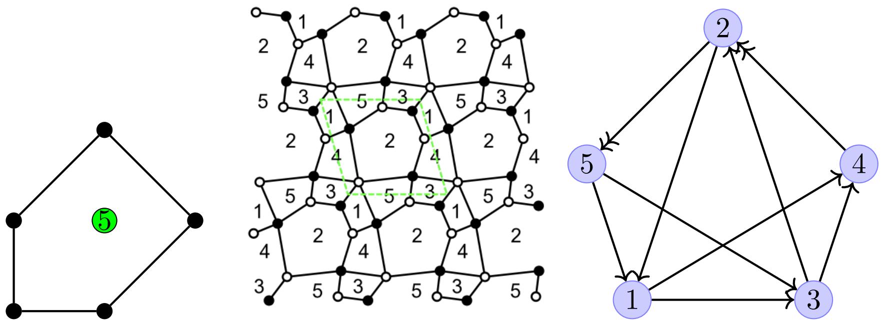

Figure 4.61. *Toric diagram, dimer and quiver for phase a of model 12.*

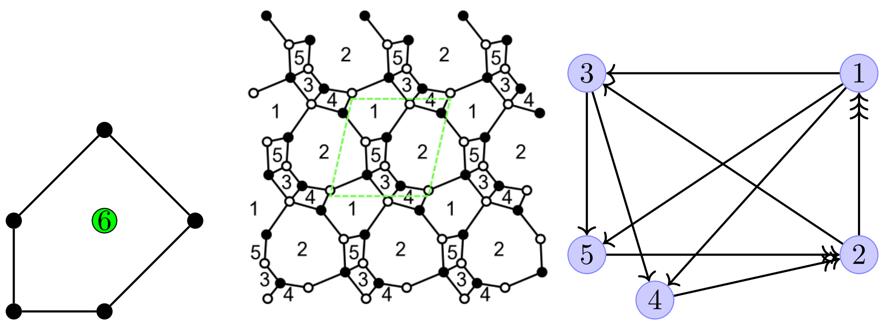

Figure 4.62. *Toric diagram, dimer and quiver for phase b of model 12.*

Model 13: cone over $Y^{2,2}$

This model has only one toric phase with twelve fields and its superpotential containing eight terms is

$$\begin{aligned} W = Tr\big(&+ X^1_{12}X_{24}X^1_{41} + X_{31}X^2_{12}X^2_{23} + X^2_{41}X_{13}X^1_{34} + X^2_{34}X_{42}X^1_{23} + \\ &- X^1_{12}X^1_{23}X_{31} - X_{13}X^2_{34}X^1_{41} - X^2_{41}X^2_{12}X_{24} - X^1_{34}X_{42}X^2_{23}\big). \end{aligned} \tag{4.88}$$

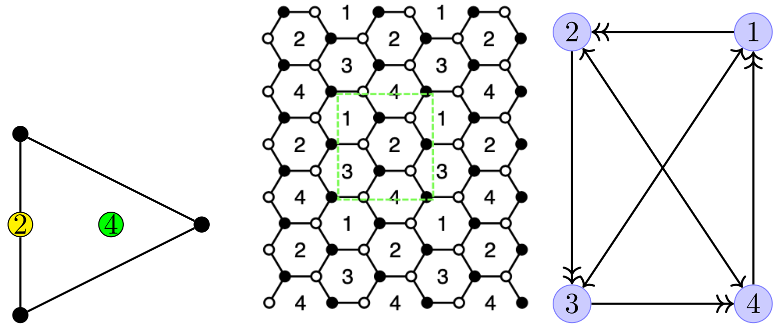

Figure 4.63. *Toric diagram, dimer and quiver for the only toric phase of model 13.*

Model 14: cone over $\mathbf{dP}_1$

Model 14 has one toric phase with ten fields, the superpotential contains six terms and it is given by

$$\begin{aligned} W = Tr\big(&+ X^1_{21}X_{14}X^1_{42} + X^3_{21}X^2_{13}X_{32} + X^2_{42}X^2_{21}X^1_{13}X_{34} + \\ &- X^1_{13}X_{32}X^1_{21} - X_{14}X^2_{42}X^3_{21} - X^2_{21}X^2_{13}X_{34}X^1_{42}\big) \end{aligned} \tag{4.89}$$

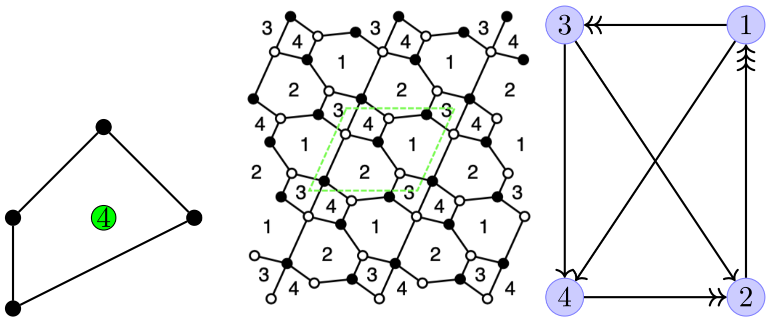

Figure 4.64. *Toric diagram, dimer and quiver for the only toric phase of model 14.*

Model 15: cone over $\mathbb{F}_0$

We have two toric phases and superpotentials given by

$$W_a = Tr\big(+ X^1_{12}X^1_{23}X^2_{34}X^2_{41} + X^2_{12}X^2_{23}X^1_{34}X^1_{41} - X^1_{12}X^2_{23}X^2_{34}X^1_{41} - X^2_{12}X^1_{23}X^1_{34}X^2_{41}\big); \tag{4.90}$$

$$W_b = Tr\big(+ X^1_{21}X^1_{14}X^1_{42} + X^2_{21}X^2_{14}X^2_{42} + X^1_{23}X^2_{34}X^3_{42} + X^2_{23}X^1_{34}X^4_{42} + \\ - X^1_{21}X^2_{14}X^3_{42} - X^2_{21}X^1_{14}X^4_{42} - X^1_{23}X^1_{34}X^2_{42} - X^2_{23}X^2_{34}X^1_{42}\big). \tag{4.91}$$

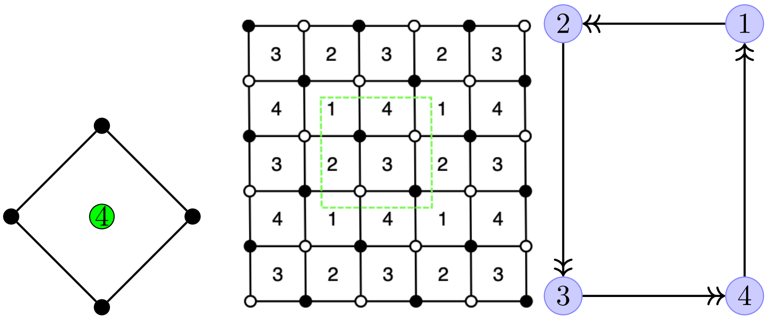

Figure 4.65. *Toric diagram, dimer and quiver for phase a of model 15.*

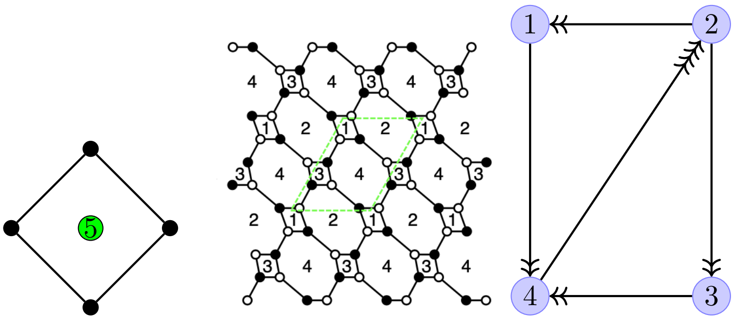

Figure 4.66. *Toric diagram, dimer and quiver for phase b of model 15.*

Model 16: cone over dP$_0$

This last model has only one toric phase with nine fields and six superpotential terms; the superpotential is given by

$$W = Tr\big(+ X^1_{12}X^3_{23}X^2_{31} + X^2_{12}X^1_{23}X^3_{31} + X^3_{12}X^2_{23}X^1_{31} + \\ - X^1_{12}X^1_{23}X^1_{31} - X^3_{12}X^3_{23}X^3_{31} - X^2_{12}X^2_{23}X^2_{31}\big). \tag{4.92}$$

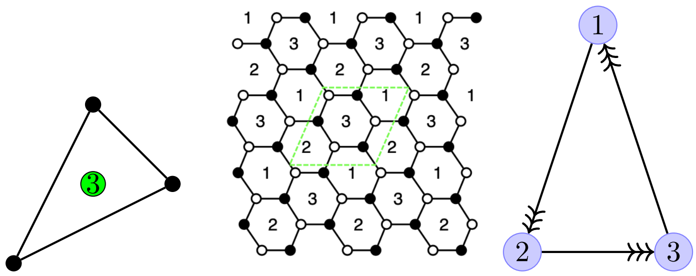

Figure 4.67. *Toric diagram, dimer and quiver for the only toric phase of model 16.*

In the following figure is concisely reported the entire classification.

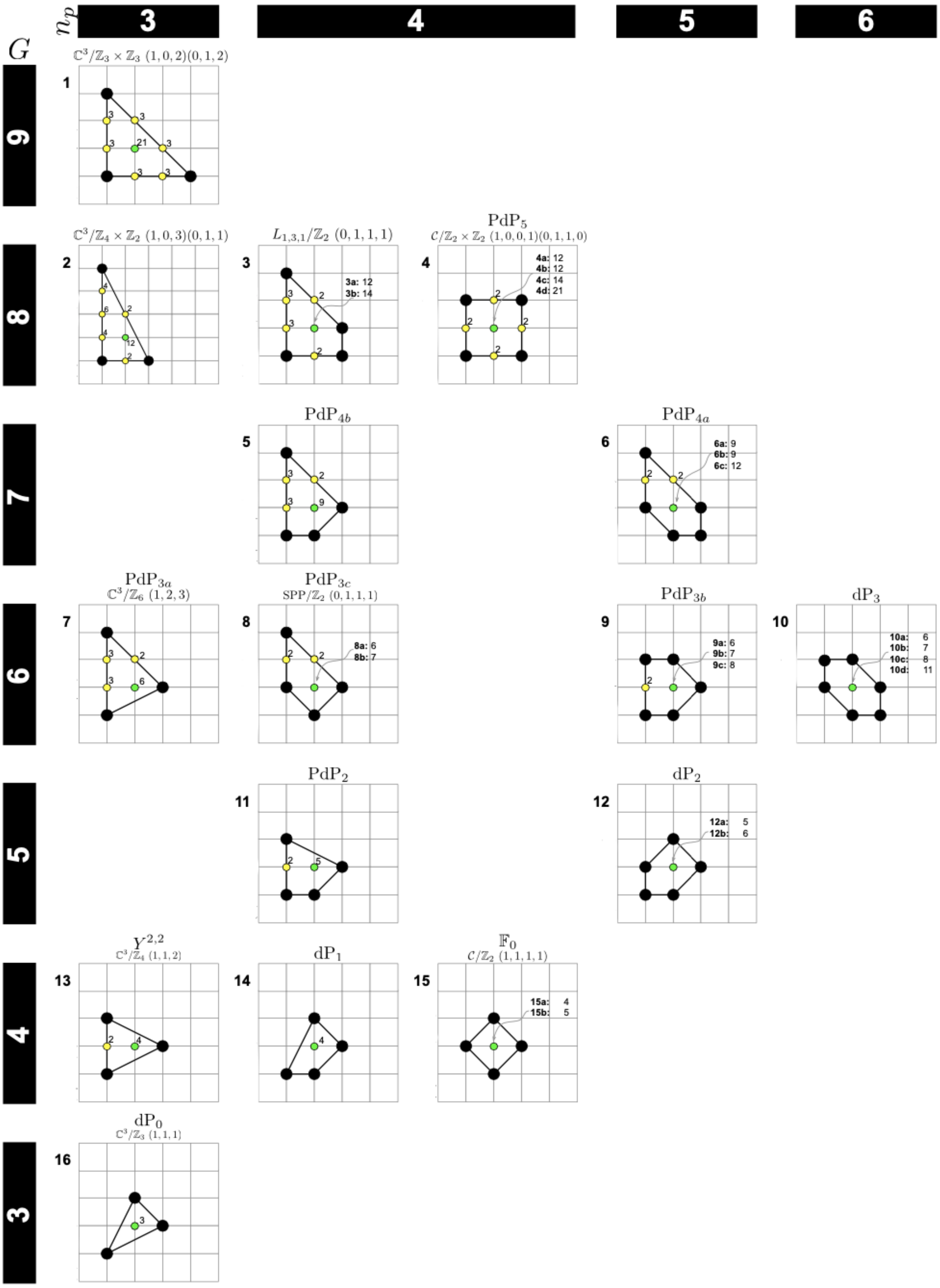

Figure 4.68. *Complete classification of the 30 reflexive polygon quiver theories. Numbers close to green and yellow points indicate their multiplicity. Arrows indicate different multiplicity of the origin points for toric dual theories. Vertex points are enumerated horizontally by n_p while the area of the polygons is given vertically by G where the area is calculated considering the smallest lattice triangle having area $G = 1$. Figure taken from [85].*

4.4 Central charge from triangles and the structure equations

In [119] Butti and Zaffaroni showed how a-maximization can be performed by considering a point $\vec{B} = (x, y)$ inside the polygon representing the toric diagram. In other words, the superconformal R-charges of a gauge theory associated to a toric geometry are determined by the point $\vec{B}$ and toric data. The procedure is given in the following.
First, we define the product between two two-dimensional vectors as

$$\langle u, v \rangle := \det \begin{pmatrix} u^{(1)} & u^{(2)} \\ v^{(1)} & v^{(2)} \end{pmatrix} . \tag{4.93}$$

For each extremal point in the toric diagram, we associate a vector v_i going from vertex i to vertex $i+1$, with $i = 1, \ldots, d \; mod(d)$ where d is the number of extremal points. The vectors w_i orthogonal to the v_i define the (p, q)-web diagram, so the product $\langle v_i, v_j \rangle$ gives the entries in the adjacency matrix. For example, if $\langle v_i, v_j \rangle = 2$ there are two fields connecting associated nodes in the quiver.
The next step is to define a set C, made by all positive $\langle v_i, v_j \rangle$. These are given by ordered pairs of vectors $(v_i,\, v_j)$ such that the associated (p, q)-web diagram vector w_i is rotated counterclockwise to w_j by an angle smaller than π.
At this point, to each vertex we associate a trial R-charge a_i and to each element $(v_i,\, v_j)$ in the set C we associate the trial R-charges combination $a_{i+1} + \ldots + a_j$. This has a pictorial intepretation at the toric diagram level: moving a (p, q)-web vector w_i to w_j, vertices from $i+1$ to j are enclosed and so one picks up their trial charge. For example, if $\langle v_1, v_3 \rangle = 2$, to the two fields a trial R-charge $a_2 + a_3$ is given. As we know, the trial charges must satisfy the condition

$$\sum_{i=1}^{d} a_i = 2 . \tag{4.94}$$

The final step is to construct the quantity

$$a = \frac{9}{32} \left[A_P + \sum_{(i,j) \in C} \langle v_i, v_j \rangle \left(a_{i+1} + \ldots + a_j - 1 \right)^3 \right] , \tag{4.95}$$

where A_P is the area of the polygon, and we must maximize it over the independent charges a_i.

This procedure gives us a way to count fields from toric data and associate them a trial R-charge, so we have done nothing but construct the central charge a from this information. In principle this is a maximization that can involve more than two variables. However, Butti and Zaffaroni give an ansatz for each a_i so that maximization of (4.95) corresponds to minimization of the volume of the Sasaki-Einstein associated to the given toric geometry, hence reducing the number of variables down to two: the coordinates (x, y) of a point $\vec{B}$ inside the toric diagram, which are nothing but the projection of the Reeb vector $\vec{b} = 3(1, x, y)$ over the polygon. They proposed that to each vertex of the polygon, we must associate another vector r_i that connects a point (x, y) inside the toric diagram to the vertex i. Then, we define the quantity

$$l_i(x,y) := \frac{\langle v_{i-1}, v_i\rangle}{\langle r_{i-1}, v_{i-1}\rangle\langle r_i, v_i\rangle} \tag{4.96}$$

and we write the trial charges as functions of (x, y)

$$a_i(x,y) = 2\frac{l_i(x,y)}{\sum_{j=1}^{d} l_j(x,y)} \ . \tag{4.97}$$

Inserting (4.97) into (4.95) we get the central charge a in terms of the two coordinates of the Reeb vector on the toric diagram. Maximizing the central charge yields $(\bar{x}, \bar{y})$ such that the theory is superconformal.

Let us now discuss what happens in case the polygon has some non extremal points[21]. We denote such points as q_i and we associate to them some trial R-charge b_i. Recall that v_i are defined such that they connect two successive extremal points, so in case there are some non extremal q_i, v_i just pass over them to reach v_{i+1}, see Fig. 4.69. For a side with q_i, there are more than one vector w of the (p, q)-web, but all of them are parallel. Suppose, as in Fig. 4.69, that a side of the polygon has vertices i and $i + 1$ with a q in the middle. In moving a vector w_{i-1} to w_i we have two choices, namely stopping before or after the point q. These choices correspond to fields with trial charges a_i and $a_i + b_i$, so we can assign to all field a trial charge with the condition

$$\sum_{i=1}^{d} a_i + \sum_{j=1}^{\bar{d}} b_j = 2 \ , \tag{4.98}$$

where $\bar{d}$ is the number of not extremal points. In [119], it is pointed out that a-maximization sets all $b_i = 0$, so Butti and Zaffaroni suggest that not extremal points are not relevant in determining the superconformal point of toric theories[22].

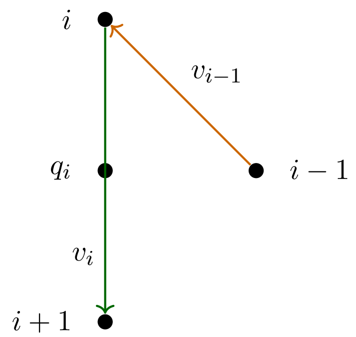

Figure 4.69. An example of a side of a polygon with a not extremal point. When we calculate the product $\langle v_{i-1}, v_i\rangle$ we ignore the not extremal point q_i but we can assign two different trial R-charges: a_i and $a_i + b_i$.

[21] Recall that this would be a singular geometry.
[22] This point has probably a more exhaustive and deeper geometrical meaning and justification.

Finally, let us note that we can triangulate the polygon using the point (x, y) as a vertex. The area of each triangle is a function of the coordinates (x, y) and it turns out, from some examples calculated explicitly, that all $a_i(x, y)$ can be expressed in terms of the triangles' area. For some theories global symmetry are present and this determines conditions that sets some trial R-charge a_i equal; this is done by knowing the perfect matchings corresponding to each point in the toric diagram and which fields they contain. Since R-charges are linked to triangles' areas, these conditions put some triangles' areas equal and set a locus on which the superconformal point (x, y) must belong to.

The fact that R-charges can be rewritten in terms of triangles' areas is a central consideration for the new extension of the Butti-Zaffaroni's work that will be presented in the following Sections. The importance of triangles' areas is largely motivated from the essence of toric geometry: its combinatoric nature. The possibility to express field theory side quantities in terms of simple combinations of geometry side quantities, such the triangles' areas, give them the status of object to be studied.

4.4.1 A rule for constructing the trial R-charges a_i from areas A_j

Let us give the rule to build up all the trial R-charges a_i in terms of the areas A_j of subtriangles constructed inside the toric diagram using the internal point $\vec{B} = (x, y)$ as a vertex for the subtriangles and v_i as the base. All expressions are of the form

$$a_i = \frac{\langle v_{i-1}, v_i \rangle}{D} \prod_{q \neq i,\, i-1} (2A_j) \ , \tag{4.99}$$

where the product involves all areas A_j which do not have v_i nor v_{i-1} as an edge, D is a combination of all the areas A_j. To find the expression for D we use the fact that $\sum_k a_k = 2$:

$$\sum_k \frac{1}{D} \langle v_{k-1}, v_k \rangle \prod_{q \neq k,\, k-1} (2A_q) = 2 \ , \tag{4.100}$$

which gives

$$D = \frac{1}{2} \sum_k \langle v_{k-1}, v_k \rangle \prod_{q \neq k,\, k-1} (2A_q) \ . \tag{4.101}$$

Plugging it back we get

$$a_i = 2 \frac{\langle v_{i-1}, v_i \rangle \prod_{q \neq i,\, i-1} (2A_q)}{\sum_k \langle v_{k-1}, v_k \rangle \prod_{q \neq k,\, k-1} (2A_q)} \tag{4.102}$$

This can be thought as a Butti-Zaffaroni's work reinterpretation in term of the triangles' areas.

4.4.2 Areas from projected Reeb vector and R-charges as projected Reeb vector work

A result that must be pointed out is the link between the areas and the work integral of the projected Reeb vector $\vec{B}$. From the point $\vec{B} = (x, y)$ we can write down an expression for polygons' area and for triangles' areas.
Let us call the polygon P and consider its area A_P written in term of the area two-form $\alpha = dx \wedge dy$:

$$A = \int_{\Omega} \alpha, \tag{4.103}$$

where $\Omega = \{(x, y) \in P\}$. From α we can find a one-form ω such that $d\omega = \alpha$; it is simple to show that $\omega = \frac{xdy - ydx}{2}$. At this one-form is associated a vector field with component $\frac{1}{2}(-y, x)$ and we note that this is the point B after the transformation belonging to $SL(2, \mathbb{Z})$ given by the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{4.104}$$

Hence polygon's area is given by

$$A_P = \int_{\Omega} \alpha = \int_{\Omega} d\omega \underbrace{=}_{Stokes} \int_{\partial\Omega} \omega, \tag{4.105}$$

where $\partial\Omega$ is the boundary of the polygon and so its edges. Splitting the integral over $\partial\Omega$ in a sum of integral over each edge S_i, we get

$$A_P = \sum_i \int_{S_i} \frac{1}{2}(xdy - ydx) = \frac{1}{2} \sum_i \int_{S_i} (-y, x) \cdot (dx, dy) = \frac{1}{2} \sum_i \int_{S_i} \vec{B} \cdot \vec{dl}. \tag{4.106}$$

At the same manner, we can write the area of a single triangle, A_j, using the projected Reeb vector:

$$A_j = \frac{1}{2} \sum_i \int_{s_i} \vec{B} \cdot \vec{dl}, \tag{4.107}$$

where now s_i are the edges of the triangle.
Moreover, we know that every $a_i(x, y)$ can be expressed in terms of the triangles' areas and so the a_i are intimately related to the projected Reeb vector' work integral:

$$a_i = 2 \frac{\langle v_{i-1}, v_i \rangle \prod_{j \neq i,\, i-1} (\int_{\gamma_j} \vec{B} \cdot \vec{dl})}{\sum_k [\langle v_{k-1}, v_k \rangle \prod_{q \neq k,\, k-1} (\int_{\gamma_q} \vec{B} \cdot \vec{dl})]}. \tag{4.108}$$

At the moment, the real meaning and the interpretation of the link between a_i and the projected Reeb vector' work integral, made explicit by 4.108, is a puzzle to be solve.
Finally, we note that since $\vec{B}$ is a not conservative field, put to zero a a_i means not to consider the triangle which has as one of the points on the perimeter of the toric, the point associated with a_i.

4.4.3 Structure equations

We now can rewrite the central charge only in term of triangles' areas:

$$a = \frac{9}{32}\Big[A_P + \sum_{(i,j)\in C} \langle v_i, v_j\rangle \Big(2\frac{\sum_{s=i+1}^{j}\langle v_{s-1}, v_s\rangle \prod_{q\neq s,\, s-1}(2A_q)}{\sum_k \langle v_{k-1}, v_k\rangle \prod_{q\neq k,\, k-1}(2A_q)} - 1\Big)^3\Big], \tag{4.109}$$

and now since we know that a is a function of x and y, we can impose the maximizing conditions: $\frac{\partial a}{\partial x} = \frac{\partial a}{\partial y} = 0$; $\frac{\partial^2 a}{\partial x^2} < 0$ and $det(H) > 0$ to derive equations for the areas that we call structure equations. Let us start with the first derivative with respect to x, the one with respect to y follows in an analogue way;

$$\begin{aligned} 0 = \frac{\partial a}{\partial x} = \frac{9}{32}\sum_{(i,j)\in C}\Big[3\langle v_i, v_j\rangle \Big(2\frac{\sum_{s=i+1}^{j}\langle v_{s-1}, v_s\rangle \prod_{q\neq s,\, s-1}(2A_q)}{\sum_k \langle v_{k-1}, v_k\rangle \prod_{q\neq k,\, k-1}(2A_q)} - 1\Big)^2 \times \\ 2\Big(\frac{[\sum_{s=i+1}^{j}\langle v_{s-1}, v_s\rangle \partial_x(\prod_{q\neq s,\, s-1}(2A_q))][\sum_k \langle v_{k-1}, v_k\rangle \prod_{q\neq k,\, k-1}(2A_q)]}{(\sum_k \langle v_{k-1}, v_k\rangle \prod_{q\neq k,\, k-1}(2A_q))^2} + \\ - \frac{[\sum_k \langle v_{k-1}, v_k\rangle \partial_x(\prod_{q\neq k,\, k-1}(2A_q))][\sum_{s=i+1}^{j}\langle v_{s-1}, v_s\rangle \prod_{q\neq s,\, s-1}(2A_q)]}{(\sum_k \langle v_{k-1}, v_k\rangle \prod_{q\neq k,\, k-1}(2A_q))^2}\Big)\Big]. \end{aligned} \tag{4.110}$$

In the following, to lighten notation, we will use $\sum_s \partial_x$ to indicate the term $\sum_{s=i+1}^{j}\langle v_{s-1}, v_s\rangle \partial_x(\prod_{q\neq s,\, s-1}(2A_q))$, $\sum_s$ to indicate $\sum_{s=i+1}^{j}\langle v_{s-1}, v_s\rangle \prod_{q\neq s,\, s-1}(2A_q)$ and similar scriptures will be used for sums over k and for the derivative with respect to y.

Let us now consider the second derivatives. The not mixed derivatives are

$$\begin{aligned} \frac{\partial^2 a}{\partial x^2} = \frac{9}{32}\sum_{(i,j)\in C}\Big\{\Big[3\langle v_i, v_j\rangle 2\Big(2\frac{\sum_s}{\sum_k} - 1\Big)2\Big(\frac{(\sum_s \partial_x)(\sum_k) - (\sum_k \partial_x)(\sum_s)}{(\sum_k)^2}\Big)\Big] \times \\ \times\Big[\frac{[(\sum_s \partial_x^2)(\sum_k) + (\sum_k \partial_x)(\sum_s \partial_x) - (\sum_k \partial_x^2)(\sum_s) - (\sum_k \partial_x)(\sum_s \partial_x)](\sum_k)^2}{(\sum_k)^4} + \\ - \frac{2(\sum_k)(\sum_k \partial_x)[(\sum_s \partial_x)(\sum_k) - (\sum_k \partial_x)(\sum_s)]}{(\sum_k)^4}\Big]\Big\}; \\ \frac{\partial^2 a}{\partial y^2} = \frac{9}{32}\sum_{(i,j)\in C}\Big\{\Big[3\langle v_i, v_j\rangle 2\Big(2\frac{\sum_s}{\sum_k} - 1\Big)2\Big(\frac{(\sum_s \partial_y)(\sum_k) - (\sum_k \partial_y)(\sum_s)}{(\sum_k)^2}\Big)\Big] \times \\ \times\Big[\frac{[(\sum_s \partial_y^2)(\sum_k) + (\sum_k \partial_y)(\sum_s \partial_y) - (\sum_k \partial_y^2)(\sum_s) - (\sum_k \partial_y)(\sum_s \partial_y)](\sum_k)^2}{(\sum_k)^4} + \\ - \frac{2(\sum_k)(\sum_k \partial_y)[(\sum_s \partial_y)(\sum_k) - (\sum_k \partial_y)(\sum_s)]}{(\sum_k)^4}\Big]\Big\}; \end{aligned} \tag{4.111}$$

and the mixed derivatives are

$$\frac{\partial^2 a}{\partial x \partial y} = \frac{9}{32} \sum_{(i,j)\in C} \Bigg\{ \Bigg[3\langle v_i, v_j\rangle 2 \left(2\frac{\sum_s}{\sum_k} - 1 \right) 2 \left(\frac{(\sum_s \partial_x)(\sum_k) - (\sum_k \partial_x)(\sum_s)}{(\sum_k)^2} \right) \Bigg] \times$$
$$\times \Bigg[\frac{[(\sum_s \partial_x\partial_y)(\sum_k) + (\sum_s \partial_y)(\sum_k \partial_x) - (\sum_k \partial_x\partial_y)(\sum_s) - (\sum_k \partial_y)(\sum_s \partial_x)](\sum_k)^2}{(\sum_k)^4} +$$
$$- \frac{2(\sum_k)(\sum_k \partial_x)[(\sum_s \partial_y)(\sum_k) - (\sum_k \partial_y)(\sum_s)]}{(\sum_k)^4} \Bigg] \Bigg\};$$
$$\frac{\partial^2 a}{\partial y \partial x} = \frac{9}{32} \sum_{(i,j)\in C} \Bigg\{ \Bigg[3\langle v_i, v_j\rangle 2 \left(2\frac{\sum_s}{\sum_k} - 1 \right) 2 \left(\frac{(\sum_s \partial_y)(\sum_k) - (\sum_k \partial_y)(\sum_s)}{(\sum_k)^2} \right) \Bigg] \times$$
$$\times \Bigg[\frac{[(\sum_s \partial_y\partial_x)(\sum_k) + (\sum_s \partial_x)(\sum_k \partial_y) - (\sum_k \partial_y\partial_x)(\sum_s) - (\sum_k \partial_x)(\sum_s \partial_y)](\sum_k)^2}{(\sum_k)^4} +$$
$$- \frac{2(\sum_k)(\sum_k \partial_y)[(\sum_s \partial_x)(\sum_k) - (\sum_k \partial_x)(\sum_s)]}{(\sum_k)^4} \Bigg] \Bigg\}. \tag{4.112}$$

Since products of triangle's areas are polynomial in x and y, a is $\mathcal{C}^2$ and the Schwarz theorem imposes the constrain $\frac{\partial^2 a}{\partial x \partial y} = \frac{\partial^2 a}{\partial y \partial x}$. We now impose the maximization constrains: first derivatives equal to zero

$$\begin{aligned} \frac{\partial a}{\partial x} &= \sum_{(i,j)\in C} \Bigg[\langle v_i, v_j\rangle \left(2\frac{\sum_s}{\sum_k} - 1 \right)^2 \left((\sum_s \partial_x)(\sum_k) - (\sum_k \partial_x)(\sum_s) \right) \Bigg] = 0; \\ \frac{\partial a}{\partial y} &= \sum_{(i,j)\in C} \Bigg[\langle v_i, v_j\rangle \left(2\frac{\sum_s}{\sum_k} - 1 \right)^2 \left((\sum_s \partial_y)(\sum_k) - (\sum_k \partial_y)(\sum_s) \right) \Bigg] = 0; \end{aligned} \tag{4.113}$$

second x-derivative less than zero

$$\frac{\partial^2 a}{\partial x^2} = \sum_{(i,j)\in C} \Bigg\{ \Bigg[\langle v_i, v_j\rangle \left(2\frac{\sum_s}{\sum_k} - 1 \right) \left((\sum_s \partial_x)(\sum_k) - (\sum_k \partial_x)(\sum_s) \right) \Bigg] \times$$
$$\times \Bigg[\Big[(\sum_s \partial_x^2)(\sum_k) - (\sum_k \partial_x^2)(\sum_s) \Big] (\sum_k)^2 - 2(\sum_k)(\sum_k \partial_x) \Big[(\sum_s \partial_x)(\sum_k) - (\sum_k \partial_x)(\sum_s) \Big] \Bigg] \Bigg\} < 0 \tag{4.114}$$

and the Hessian greater than zero

$$\frac{\partial^2 a}{\partial x^2}\frac{\partial^2 a}{\partial y^2} - \left(\frac{\partial^2 a}{\partial x \partial y} \right)^2 > 0 \tag{4.115}$$

Eqs 4.113, 4.114 and 4.115 are the structure equations of the toric model and, if the triangles' areas that make the toric diagram satisfy these relations between them, then the central charge is automatically maximized. At the moment, there is no clear interpretation of these equations and likely they are too computationally expensive to be used to maximize the central charge and even in the simplest case they are terribly tricky. Nevertheless, they may give interesting constrains on the areas which would require a more in depth study in the future since they may contain information about the symmetries of the system. Moreover, we must remember that all this can be linked to the work integral of the projected Reeb vector. In the end, we guess that the structure equations can give useful information on the theory even

if, for the moment, difficult to extract.
Another type of structure equation can be found if we ask ourselves how the central charge changes when we vary one of the triangles' areas. This point is quite subtle, take the derivative of a with respect to one of the triangles' areas, say A_j, seems to be not difficult. However if we want to maintain fixed the total area A_P (remaining so on the same toric model) we need to know how the triangles' areas are linked to the variation of A_j. This problem could be approached in the future.

Let us now show two examples of how construct the trial charges from triangles' areas and how some symmetries of the theory can select the locus where the central charge is maximized.

Example 1: $\frac{\mathbb{C}^3}{\mathbb{Z}_3}$

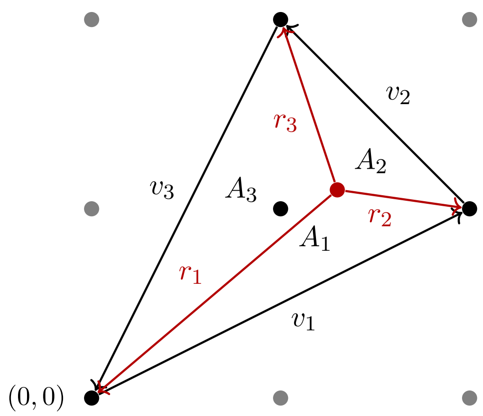

Figure 4.70. Toric diagram of $\frac{\mathbb{C}^3}{\mathbb{Z}_3}$.

$$
\begin{aligned}
v_1 &= (2,1)\,, & r_1 &= (2-x, 1-y)\,, & A_1 &= \tfrac{1}{2}(2y-x)\,;\\
v_2 &= (-1,1)\,, & r_2 &= (1-x, 2-y)\,, & A_2 &= \tfrac{1}{2}(3-x-y)\,;\\
v_3 &= (-1,-2)\,, & r_3 &= (-x,-y)\,, & A_3 &= \tfrac{1}{2}(2x-y)\,.
\end{aligned}
\tag{4.116}
$$

The set C and associated trial R-charges a_i are

$$
\begin{aligned}
\langle v_1, v_2\rangle &= 3 \to a_2\;;\\
\langle v_2, v_3\rangle &= 3 \to a_3\;;\\
\langle v_3, v_1\rangle &= 3 \to a_1\;;
\end{aligned}
$$

and using Eq. (4.97) and Eq. (4.102) we obtain

$$a_1 = \frac{2A_2}{A_1 + A_2 + A_3} = \frac{2}{3}(3 - x - y) ;$$
$$a_2 = \frac{2A_3}{A_1 + A_2 + A_3} = \frac{2}{3}(2x - y) ;$$
$$a_3 = \frac{2A_1}{A_1 + A_2 + A_3} = \frac{2}{3}(2y - x) . \tag{4.117}$$

The central charge a, from Eq. (4.95) is

$$a = \frac{9}{32}\left[3 + 3(a_2 - 1)^3 + 3(a_3 - 1)^3 + 3(a_1 - 1)^3\right] , \tag{4.118}$$

and has a maximum at $(\bar{x}, \bar{y}) = (1, 1)$. Note that this theory enjoys a $SU(3)$ global symmetry, reflected by the fact that exchanging all a_i leave the central charge invariant. Imposing that $a_1 = a_2 = a_3$ gives that $A_1 = A_2 = A_3$ and this happens exactly at $(\bar{x}, \bar{y}) = (1, 1)$. In this point the R-charges are all equal to $\frac{2}{3}$ and so all the 9 fields have the same R-charge.
This information can be obtained directly from the structure equations; the procedure is more complicated but it is quite evident that the structure equations should provide additional information on the theory beyond the $SU(3)$ symmetry and the maximum point.

Example 2: SSP

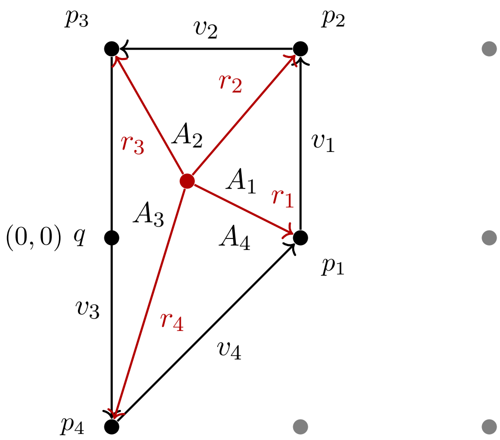

Figure 4.71. Toric diagram of SPP.

$$\begin{aligned} v_1 &= (0,1) , & r_1 &= (1-x, -y) , & A_1 &= \tfrac{1}{2}(1-x) ; \\ v_2 &= (-1,0) , & r_2 &= (1-x, 1-y) , & A_2 &= \tfrac{1}{2}(1-y) ; \\ v_3 &= (0,-2) , & r_3 &= (-x, 1-y) , & A_3 &= x ; \\ v_4 &= (1,1) , & r_4 &= (-x, -1-y) , & A_4 &= \tfrac{1}{2}(1-x+y) ; \end{aligned} \tag{4.119}$$

The set C is given by

$$\begin{aligned}
\langle v_1, v_2\rangle &= 1 \to a_2 \ ; \\
\langle v_2, v_3\rangle &= 2 \to a_3 \ ; \\
\langle v_4, v_2\rangle &= 1 \to a_1 + a_2 \ ; \\
\langle v_3, v_4\rangle &= 2 \to a_4 \ ; \\
\langle v_4, v_1\rangle &= 1 \to a_1 \ ;
\end{aligned} \tag{4.120}$$

and trial R-charges are

$$\begin{aligned}
a_1 &= \frac{A_2A_3}{A_2A_3 + A_3A_4 + 2A_4A_1 + 2A_1A_2} = \frac{2}{2-x} x\,(1-y) \ ; \\
a_2 &= \frac{A_3A_4}{A_2A_3 + A_3A_4 + 2A_4A_1 + 2A_1A_2} = \frac{2}{2-x} x\,(1-x+y) \ ; \\
a_3 &= \frac{2A_4A_1}{A_2A_3 + A_3A_4 + 2A_4A_1 + 2A_1A_2} = \frac{2}{2-x}\,(1-x)\,(1-x+y) \ ; \\
a_4 &= \frac{2A_1A_2}{A_2A_3 + A_3A_4 + 2A_4A_1 + 2A_1A_2} = \frac{2}{2-x}\,(1-x)\,(1-y) \ .
\end{aligned} \tag{4.121}$$

The central charge a, from Eq. (4.95), is

$$a = \frac{9}{32}\left[3 + (a_2-1)^3 + 2\,(a_3-1)^3 + (a_1+a_2-1)^3 + 2\,(a_4-1)^3 + (a_1-1)^3\right], \tag{4.122}$$

and has a maximum at

$$\begin{aligned}
\bar{x} &= 1 - \frac{1}{\sqrt{3}} \ ; \\
\bar{y} &= \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right) \ ;
\end{aligned} \tag{4.123}$$

where

$$\begin{aligned}
a_1 = a_2 &= 1 - \frac{1}{\sqrt{3}} \ ; \\
a_3 = a_4 &= \frac{1}{\sqrt{3}}
\end{aligned} \tag{4.124}$$

and the fields in set C have R-charges

$$\begin{aligned}
a_2 &= 1 - \frac{1}{\sqrt{3}} \; ; \\
a_3 &= \frac{1}{\sqrt{3}} \; ; \\
a_1 + a_2 &= 2\left(1 - \frac{1}{\sqrt{3}}\right) \; ; \\
a_4 &= \frac{1}{\sqrt{3}} \; ; \\
a_1 &= 1 - \frac{1}{\sqrt{3}} \; .
\end{aligned} \tag{4.125}$$

From the central charge a, we see that we can exchange $a_1 \leftrightarrow a_2$ and $a_3 \leftrightarrow a_4$. Imposing that $a_1 = a_2$ and $a_3 = a_4$ gives $A_2 = A_4$ and we get

$$y = \frac{x}{2} \; . \tag{4.126}$$

Indeed, the point that maximizes the central charge sits on this line. It seems evident that the structure equations should, among the other information, also fix the correct x.

Chapter 5

Orientifold projections and unoriented quiver theories

In previous chapters we have seen some examples of quiver gauge theories dual, thanks to extensions of AdS/CFT correspondence, to a configuration of D3-branes sitting on a conical singularity of a CY cone. We have also seen the five brane diagram algorithm that allows us to build up the quiver gauge theory knowing the toric diagram of the CY threefold geometry. Now, we want to further extend the class of quiver theories taken into account and we will do it considering the so-called orientifold projections. These are particular involution[1] of string theory that map oriented string into unoriented ones; from the point of view of space-time background these projections are due to the presence of mirror planes called Op-planes. We will see that these projections have an interpretation in the brane tiling picture as the $\mathbb{Z}_2$ symmetry of the dimer or bipartite graph and we will give set of rules that tells us how fields, gauge groups and superpotential terms behave under these orientifold projections.
First, we will see orientifold projections, briefly from the string point of view [112],[114],[115] and more in depth from the branes and dimers picture point of view [113],[116],[118]. Next, we will study some orientifold projections of the thirty reflexive polygons quiver theories. In the last section we will study how post orientifold central charge can be computed from trivial and not trivial automorphism of the toric diagram.

5.1 Orientifolds

We said that orientifold projections map oriented strings to unoriented ones, but let us see how. Denoting by $0 \leq \sigma \leq \pi$ the coordinate describing the open string at a given time, the two ends $\sigma = 0, \pi$ contain, thanks to Chan-Paton indexes, the gauge group degrees of freedom and the corresponding charged matter fields. As we know, on this endpoints we can apply Dirichlet or Neumann boundary conditions and we recall that under T-duality these are interchanged. Orientifold projections of Type IIB theory are obtained by projecting the Type IIB spectrum by the involution

[1] Recall that this means that $\Omega^2 = 1$.

Ω, exchanging the left and right closed oscillators $\hat{\alpha}^\mu_m$, $\hat{\beta}^\mu_m$ and acting on the open strings as phases:

$$\begin{aligned} &\textit{closed strings} \Rightarrow \Omega X^\mu(\tau,\sigma)\Omega^{-1} = X^\mu(\tau,-\sigma) \Rightarrow \hat{\alpha}^\mu_m \leftrightarrow \hat{\beta}^\mu_m, \\ &\textit{open strings} \Rightarrow \Omega X^\mu(\tau,\sigma)\Omega^{-1} = X^\mu(\tau,\pi-\sigma) \Rightarrow \hat{\alpha}^\mu_m \to \pm(-1)^m \hat{\alpha}^\mu_m; \end{aligned} \tag{5.1}$$

the very interesting fact [117] is that this orbifold projection of type IIB superstring theory is equal to type I superstring theory. The presence of unoriented strings change drastically the topological expansion of the string perturbation theory; indeed now are allowed also non oriented surfaces such as Klein bottle or Möbius strip. Let us be more rigorous. String perturbation theory consists of a sum over the worldsheet topologies of increasing complication, which is quantified by the Euler characteristic χ, and the amplitude associated with a given topology is proportional to $g_s^{-\chi}$. In a theory of oriented strings, the worldsheets are two dimensional complex oriented surfaces topologically classified by their genus h, and their number of boundaries b[2]. If we admit unoriented strings, the expansion must contain also unoriented surfaces with Euler characteristic given by $\chi = 2 - 2h - b - c$ where c is the number of crosscaps. A crosscap is a representation of the real projective plane: it is like a shrinked torus where there is no middle hole and the side has been clipped so that they cross. The crosscap is made of a one parameter family of circles and the strip between two neighbor circles is a Möbius strip. Below a representation of the crosscap. Polchinski realized [45] that orientifold projections have a simple and

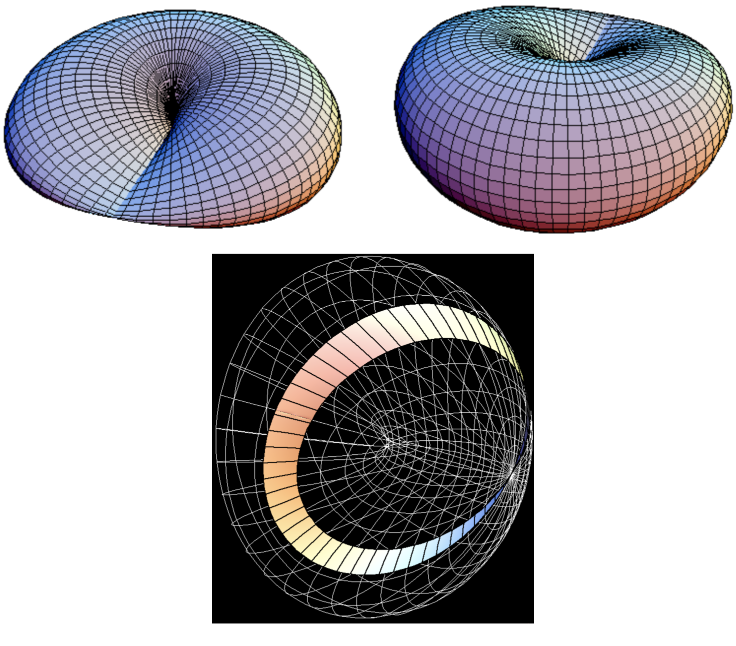

Figure 5.1. *Top panel: crosscap surface representation. Bottom panel: strip between two neighbor circles, this is a Möbius strip.*

elegant interpretation from the point of view of the background space-time: they correspond to not dynamical, mirror like objects called orientifold planes Op, defined

[2]These are Riemannian surfaces with $\chi = 2 - 2h - b$.

by T-duality as fixed points of the orientifold projections. Turns out that doing n T-dualities we get 2^n O$(9-n)$-planes and the orientifold projection act as

$$\begin{aligned} \Omega' &= \Omega\Pi_1 ... \Pi_n \qquad\qquad if\ n\ odd, \\ \Omega' &= (-1)^{\hat{F}_L}\Omega\Pi_1 ... \Pi_n\ if\ n\ even, \end{aligned} \tag{5.2}$$

where $\hat{F}_L$ is the space-time left fermion operator; Ω is the orientifold projection before the n T-dualities and Π_i are space-time parity operators. Moreover this Op-planes carry $\pm 2^{p-5}$ units of $(p+1)$-form charge if we normalize the Dp-brane charge to one units and they have negative tension given by $T_{Op} = -2^{p-5}T_{Dp}$ where T_{Dp} is the tension of a Dp-brane. These orientifold planes are the fixed loci of the orientifold projection's action on the space-time and they can be thought as generalized orbifolds in which the geometric projection is accompanied by an action on the spectum. An intuitive way to think at orietifolded theories is to view the orientifold planes as mirrors dividing the space into two halves: this is a $\mathbb{Z}_2$ projection. Whatever lies in one of the two halves has a mirror image in the other one, hence a string moving on one side does not feel the effect of the mirror plane but a string connecting two mirrored points crosses the orientifold planes and becomes unoriented.

Since we are interested in AdS/CFT correspondence we want to study what kind of theoris arise on stack of N D3-branes in the presence of orientifold planes. When the stack of branes lies in one of the two halves there is a mirror stack of branes in the other halves. The worldvolume gauge theory has gauge group $U(N)$, but besides the adjoint fields, there is matter either in the symmetric or in the antisymmetric representation depending on the sign of the R-R charge of the orientifold plane, arising from the open unoriented strings connecting the stack branes with its mirror and so crossing the orientifold plane. If instead, we consider a stack of N branes parallel and coincidental with the orientifold plane, it is mapped to itself by the orientifold. The open strings which end on these branes are also ending on the orientifold plane and the gauge degrees of freedom are projected. The gauge group is either $USp(N)$ or $SO(N)$ depending on the sign of the R-R charge of the orientifold plane. Gauge theories that arise from this brane configurations are typically generalization of quiver gauge theories and they are called unoriented quiver gauge theories. To study four dimensional unoriented quiver gauge theories, basic configuration is given by a stack of N D3-branes with some orientifold planes that share three dimensions with the D3-branes, thus being at the conical singularity of a CY cone threefold and with the remaining directions extending in transverse space. The resulting theory has closed unoriented strings moving in a $\mathbb{Z}_2$ projection of the original CY and open strings ending on the stack of D3-branes. In this context AdS/CFT correspondence gives us a duality between a four dimensional unoriented quiver gauge theory and a theory of quantum gravity given by an unoriented string theory in AdS$_5 \times \frac{X_5}{\mathbb{Z}_2}$ backgound space-time, where X_5 is the Sasaki-Einstein base of the Calabi-Yau cone. However, performing orientifold projections in CY cone singularities is not possible in general and, when it does, it is very difficult. Nevertheless, as we could imagine, toric condition on geometry makes complications due to orientifold plane more tractable. In the context of the holographic duality, formula $\chi = 2 - 2h - b - c$ assumes a new interpretation: in the large N limit the effect on the gauge theory of the orientifold is negligible. This is because the gravitational theory dual to an unoriented quiver gauge theory lives in a $\mathbb{Z}_2$ projected space and contains unoriented strings; their string coupling g_s is inversely proportional to the large N limit rank N of the gauge group. Hence we expect that the observables of the gauge theory, which can be computed by string perturbation theory for small g_s, are the same as they would be if the dual string theory were an orbifold projection. However, this argument is too naive and turns out to be not true in general since the introduction of orientifold planes can make certain supergravity solutions unstable and turns out that finite N, or not zero g_s, effects are still important in the large N limit or in the limit $g_s \to 0$. What is certainly true is that term with not zero genus, boundaries or crosscaps are subleading terms of the string perturbative expansion.

5.1.1 Five brane system point of view

We know that all gauge theory information are contained into the bipartite graph or the dimer, and we have discussed how this emerge from T-duality of the usual stack of N D3-branes sitting on the conical singularity of a CY cone. Now, we want to understand how we can represent graphically orientifold projections.
Under T-duality Op-planes behave exactly as Dp-branes: they are mapped to other O-planes of different dimensionality. Hence, if the CY geometry allows for an orientifold projection, the latter becomes manifest itself as a $\mathbb{Z}_2$ involution on the bipartite graph or dimer and the orientifold fixed loci are the intersections of the O-planes with the fundamental cell[3]. The three possible $\mathbb{Z}_2$ involutions of topological torus are drown below; we have the so-called fixed points projection, the fixed line and lines projections. In the following we will write "fixed line(s)" to indicate both b and c orientifold projections. These $\mathbb{Z}_2$ projections of bipartite graph or dimer means that every object is mapped or to itself or to a mirror object.

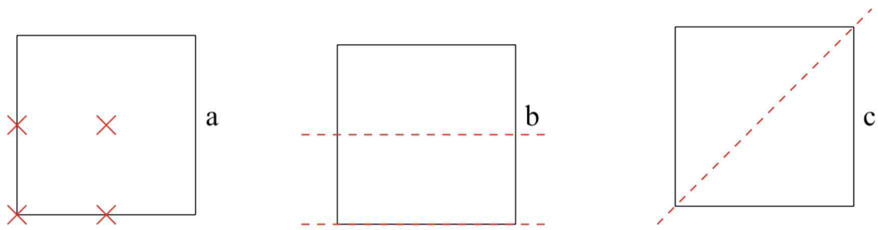

Figure 5.2. *Possible $\mathbb{Z}_2$ involutions of topological torus:* a *has four fixed points,* b *has two fixed lines and* c *has one fixed line.*

Fixed points and fixed line(s) projections differ for how they act on NS5 branes: the first preserve their orientation, while the seconds must reverse the orientations. These contrains have useful consequences:

- from a bipartite graph point of view, fixed points orientifolds must map each circle to a circle of the opposite color while fixed-line(s) orientifolds map circles to circles of the same color;

[3]Recall that the fundamental cell is a topological torus.

- from a dimer point of view, fixed points orientifolds must map each vertex to a vertex of the opposite color while fixed-line(s) orientifolds map vertices to vertices of the same color.

As we said before, an important role is played by the charge of orientifold planes. In the bipartite graph and dimer, the O-planes are the fixed loci and they must carry a sign, that we denote by $\epsilon = \pm$, and that indicates the positive or negative R-R charge of the correspondent orientifold plane. Moreover, remembering that objects in bipartite graph or dimer correspond to fields and gauge groups, we can look at orientifold projections action on them to understand which and how objects are mapped under orientifolds. Indeed, there is a set of rules [113]:

- given a gauge group a, if the mirror gauge group $\tilde{a}$ is different from a, they are identified and this is an unitary group. A fundamental representation of a corresponds to an antifundamental representation of $\tilde{a}$ while an antifundamental representation of a corresponds to a fundamental representation of $\tilde{a}$;

- given a field X_{ab} transforming as $(\square_a, \bar{\square}_b)$ with respect to gauge groups a and b, if the mirror field $\tilde{X}_{\tilde{b}\tilde{a}}$ transforming as $(\bar{\square}_{\tilde{a}}, \square_{\tilde{b}})$ with respect to gauge groups $\tilde{a}$ and $\tilde{b}$ is different from X_{ab}, they are identified;

- given a field $X_{a\tilde{a}}$ transforming as $(\square_a, \bar{\square}_{\tilde{a}})$ with respect to gauge groups a and its mirror $\tilde{a}$ then $(\square_a, \bar{\square}_{\tilde{a}}) = (\square_a, \square_a)$; this is not a irreducible representation and so it is decomposed in the direct sum of symmetric and antisymmetric representations;

- given a gauge group a, if the mirror gauge group is itself, this is projected to a symplectic group if $\epsilon = -$ or to an othogonal group if $\epsilon = +$;

- given a field X_{ab} transforming as $(\square_a, \bar{\square}_b)$ with respect to gauge groups a and b, if the mirror field is itself ,its representation is the symmetric one if $\epsilon = +$ and the antisymmetric one if $\epsilon = -$.

Now we have to discuss the superpotential of the unoriented theory. We remark that the superpotential is what makes the mesonic moduli space equal to the Calabi-Yau probed by the D3-branes configuration. In general, after the orientifold projection, the geometry is a $\mathbb{Z}_2$ projection of the parent Calabi-Yau; therefore, we expect that the unoriented superpotential is some kind of projection of the parent one. Infact the superpotential is obtained from the parent one by keeping only one term for each couple of identified vertices or circles. In the fixed line(s) case, where a vertex may lie on the top of an O-plane, this vertex turns out to be present in the superpotential of the son theory.

We have seen that the sign ϵ of the fixed loci is important to understand what happens to gauge group and fields, hence is interesting to better clarify it. In case of fixed line(s) every line can have $\epsilon = \pm$ while in the case of fixed points there is a rule that set the overall sign:

$$\prod_{i=1}^{4} \epsilon_i = (-1)^{\frac{N_W}{2}}, \tag{5.3}$$

where N_W is the number of superpotential terms of the parent theory and ϵ_i is the sign of the ith fixed point.

Although this set of rules makes possible for us to understand what effects orientifolds have on fields gauge groups and superpotential, it remains not clear how to understand, in general, whether a given geometry admits orientifolds or not only looking at its toric diagram.

5.1.2 Standard lore and beyond

We have already saw that in string theory is possible to consider the orientifold planes which induce a $\mathbb{Z}_2$ involution on the space-time and make strings unoriented. On the gauge theory side, as we know, this results in more general gauge theories in which is allowed the presence of orthogonal and symplectic groups as well as matter content in symmetric and antisymmetric representations. Moreover, the presence of orientifold planes modifies the RG flow and two different scenarios have been investigated so far in the literature and considered as the standard lore:

- in the I scenario there is a superconformal fixed point, and the R-charges of the operators that are not projected out by Ω are the same as the charges of the corresponding oriented theory in the large N limit. In the end, the post-orientifold central charge a^{Ω} turns out to be half of the pre-orientifold central charge a;
- II scenario does not admit a RG fixed point.

However, last year [120], based on the explicit example of PdP_{3c} and PdP_{3b}, a new possibility was proposed: the III scenario. In this case $\frac{a^{\Omega_2}}{a} < \frac{1}{2}$. Surprisingly, the analysis of [120], partially reported in Appendix D, shows that the values of $a^{\Omega_2}_{\text{PdP}_{3c}}$ and the R-charges of this theory coincide, for any N, with the ones of the unoriented theory associated to PdP_{3b} that realizes, instead, the I scenario. In the standard lore, the orientifold projection is believed to modify the R-charges only at subleading orders. However, in the specific model studied in [120] these subleading corrections break the superconformal symmetry of the PdP_{3c} parent theory and the fact that a maximization gives a new RG fixed point suggests that the theory flows to a new superconformal fixed point in the IR. In this sense, the III scenario suggests the possibility of an IR duality between two orientifold theories and seems to be a novel possibility not considered before which is natural to investigate in deeper detail since for the moment there is no explanation on when the III scenario can occur.
Other examples were found a few months after the third scenario was proposed [121]. Anyhow, the theories considered in [121] are non chiral orbifold of SPP model, $\frac{\text{SPP}}{\mathbb{Z}'_n}$; the orientifolds of these theories are in III scenario with $L^{(\frac{3n}{2},\frac{3n}{2},\frac{3n}{2})}$ for n even and with $L^{(\frac{3n-1}{2},\frac{3n+1}{2},\frac{3n-1}{2})}$ for n odd. In the end, the case of PdP_{3c} and PdP_{3b} is the only one, at the moment, that contains chiral theories in III scenario.
At the moment it is far from to be clear when a theory admits orientifold in III scenario and, if the III scenario is admitted, with what other theory is realized the IR duality. Moreover, there are examples of theories that admit more then one

possible orientifold and happens that some of them are in I or II scenario while other in III scenario; also this, is a puzzle.

5.2 Towards a classification of the orientifolds of reflexive polygon quiver theories and III scenario orientifolds

Let us now study in a certain detail the orientifold projections of the reflexive polygon quiver theories. We restrict ourselves to models that have no different toric phases and that are related by a blow up or a blow down. The analysis of these orientifolds take into account the gauge anomaly cancellation condition, and we give a list of all the possible choices of signs that make orientifold not anomalous. We will write $\Omega = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ for fixed points orientifold, $\Omega = (\tau_1, \tau_2)$ for fixed lines orientifold and $\Omega = (\eta)$ in the case of fixed line orientifold

Model 16: $\frac{\mathbb{C}^3}{\mathbb{Z}_3}$

This model has fixed line and fixed points orientifolds; the quiver for the unoriented theories is drown below

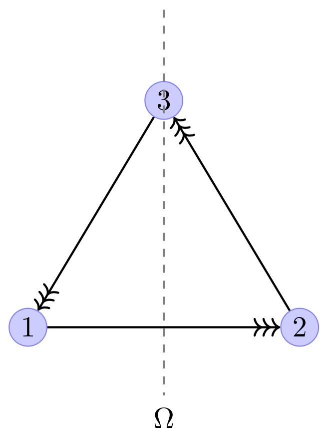

Figure 5.3. *Unoriented quiver diagram for fixed line and fixed points orientifolds of* $\frac{\mathbb{C}^3}{\mathbb{Z}_3}$.

In the case of fixed line, the orientifold projects out one gauge group (3) and one field (X^1_{12}) while X^2_{12} and X^3_{12} are identified. Moreover gauge groups 2 and 1 are also identified and so X^2_{12} (or alternatively X^3_{12}) is in a reducible representation and must be decomposed as the direct sum of antisymmetric and symmetric representations. From gauge anomaly cancellation condition for gauge group 2 we find

$$3N_3 - (N_2 + 4\eta) - (N_2 + 4\eta) - (N_2 - 4\eta) = 0 \Rightarrow 3(N_3 - N_2) = 4\eta, \tag{5.4}$$

but since $N_3 - N_2$ must be an integer number, this orientifold is anomalous.

In the case of fixed points, the orientifold projects out one gauge group (3) and three fields $(X^1_{12}, X^2_{12}, X^3_{12})$; the gauge anomaly condition for gauge group 2 is

$$3N_3 - (N_2 + 4\epsilon_1) - (N_2 + 4\epsilon_3) - (N_2 + 4\epsilon_4) = 0 \Rightarrow 3(N_3 - N_2) = 4(\epsilon_1 + \epsilon_3 + \epsilon_4) \quad (5.5)$$

and we have anomaly cancellation if $\epsilon_1 + \epsilon_3 + \epsilon_4 = \pm 3$. Hence we have or $N_3 = N_2 + 4$ or $N_3 = N_2 - 4$. Moreover we have the condition $\prod_i \epsilon_i = -1$ since the parent theory has six superpotential terms; hence the possible choice of signs are

$$\begin{aligned} \Omega_1 &= (+, -, +, +); \\ \Omega_2 &= (-, +, -, -). \end{aligned} \quad (5.6)$$

Model 14: cone over dP$_1$

This model admits only fixed points orientifold; the unoriented quiver is reported below.

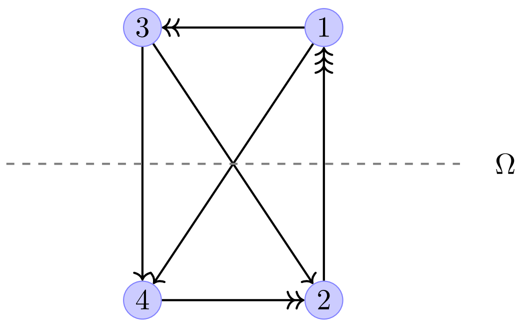

Figure 5.4. *Unoriented quiver diagram for fixed points orientifold of dP$_1$.*

The orientifold projects out four fields $(X_{34}, X^1_{21}, X^2_{21}, X^3_{21})$ and the gauge anomaly cancellation conditions are

$$\begin{aligned} 1 &: 2N_3 - (N_1 + 4\epsilon_1) - (N_1 + 4\epsilon_2) - (N_1 + 4\epsilon_3) + N_3 = 0; \\ 2 &: (N_1 + 4\epsilon_1) + (N_1 + 4\epsilon_2) + (N_1 + 4\epsilon_3) - 2N_3 - N_3 = 0; \end{aligned} \quad (5.7)$$

which are not independent and require, since $N_1 - N_3$ must be an integer,

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = \pm 3. \quad (5.8)$$

Hence ranks must satisfy $N_1 = N_3 - 4$ or $N_1 = N_3 + 4$. Since $\prod_i \epsilon_i = -1$ we have the two possibilities

$$\begin{aligned} \Omega_1 &= (+, +, +, -); \\ \Omega_2 &= (-, -, -, +). \end{aligned} \quad (5.9)$$

Model 13: cone over $Y^{2,2}$

This theory admits fixed lines and fixed points orientifold; the unoriented quiver for both orientifold type are shown below.
The fixed points orientifold project out two gauge groups (2 and 4) and two fields (X_{31}, X_{13}) while the fixed lines one project out the same gauge groups (2 and 4)

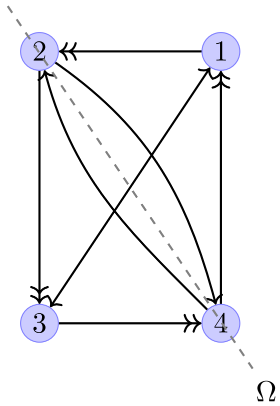

Figure 5.5. *Unoriented quiver diagram for both fixed lines or fixed points orientifold of* $Y^{2,2}$.

and the same fields (X_{31}, X_{13}). The gauge anomaly free condition for fixed points is given by

$$2N_2 - 2N_4 + (N_3 + 4\epsilon_1) - (N_3 + 4\epsilon_2) = 0 \Rightarrow N_2 - N_4 = 2(\epsilon_2 - \epsilon_1), \tag{5.10}$$

which gives constrains on the gauge groups ranks:

$$\begin{aligned} N_2 - N_4 &= 0 \quad && if\ \epsilon_1 = \epsilon_2; \\ N_2 - N_4 &= 4\epsilon_1 \quad && if\ \epsilon_1 = -\epsilon_2. \end{aligned} \tag{5.11}$$

Hence we have no constrains on ϵ_1 and ϵ_2; however we know that the overall sign must be + and the possible choices are:

$$\begin{aligned} \Omega_{1,2} &= (+, +, \pm, \pm); \\ \Omega_{3,4} &= (-, -, \pm, \pm); \\ \Omega_{5,6} &= (+, -, \pm, \mp); \\ \Omega_{7,8} &= (-, +, \pm, \mp). \end{aligned} \tag{5.12}$$

The case of fixed lines is quite similar: gauge anomaly cancellation condition gives us the same information of the previous case but now we have only four possible choices

$$\begin{aligned} \Omega_1 &= (+, +); \\ \Omega_2 &= (-, -); \\ \Omega_3 &= (+, -); \\ \Omega_4 &= (-, +). \end{aligned} \tag{5.13}$$

Model 11: cone over $\mathbf{PdP}_2$

This model admits only fixed points orientifold in which one gauge group (5) and three fields (X_{12}, X_{21}, X_{34}) are projected out; the unoriented quiver is given by the figure below The anomaly free conditions imposes:

$$\begin{aligned} &2 : 2N_2 - N_4 + (N_2 + 4\epsilon_1) - (N_2 + 4\epsilon_2) - N_4 = 0; \\ &4 : N_4 - (N_4 + 4\epsilon_3) - N_2 + N_5 = 0; \end{aligned} \tag{5.14}$$

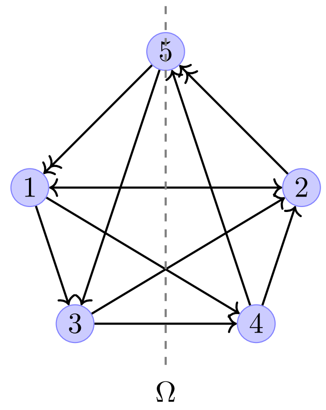

Figure 5.6. *Unoriented quiver diagram for the fixed points orientifold of the model PdP_2.*

and so

$$N_5 - N_2 = 4\epsilon_3 \ , \ N_5 - N_4 = 4\epsilon_3 - 2(\epsilon_1 - \epsilon_2). \tag{5.15}$$

The possibles signs choices are

$$\begin{aligned} \Omega_{1,2} &= (+,+,\pm,\pm); \\ \Omega_{3,4} &= (-,-,\pm,\pm); \\ \Omega_{5,6} &= (+,-,\pm,\mp); \\ \Omega_{7,8} &= (-,+,\pm,\mp). \end{aligned} \tag{5.16}$$

since the overall sign must be +.

Model 7: cone over $\mathbf{PdP}_{3a}$

This model has fixed points orientifold and the unoriented quiver is represented below

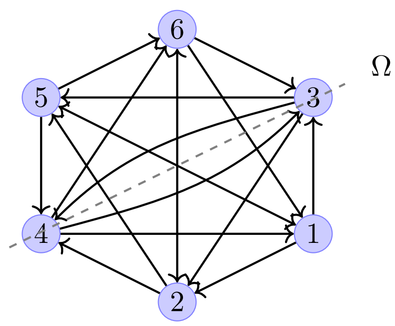

Figure 5.7. *Unoriented quiver for fixed points orientifold of PdP_{3a}.*

The orientifold projects out two gauge groups (3 and 4) and two fields (X_{61}, X_{52}); anomaly free conditions are

$$\begin{aligned} 1 &: N_3 - (N_1 + 4\epsilon_1) + N_2 - N_2 - N_4 + N_1 = 0; \\ 2 &: N_4 + (N_2 + 4\epsilon_2) + N_1 - N_1 - N_3 - N_1 = 0. \end{aligned} \tag{5.17}$$

Hence we have, adding the two equations

$$-N_1 - 4\epsilon_1 + N_2 + 4\epsilon_2 = 0 \Rightarrow N_2 - N_1 = 4(\epsilon_1 - \epsilon_2) \tag{5.18}$$

while subtracting them and using the equation above we get

$$\begin{aligned} &N_3 - N_1 - 4\epsilon_1 - N_4 + N_1 - N_4 - N_2 - 4\epsilon_2 + N_3 + N_1 = 0 \Rightarrow \\ &2N_3 + N_1 - 4(\epsilon_1 + \epsilon_2) - 2N_4 - N_2 = 0 \Rightarrow \\ &2N_3 - 8\epsilon_1 - 2N_4 \Rightarrow N_3 - N_4 = 4\epsilon_1. \end{aligned} \tag{5.19}$$

The possible signs choices are constrained only by the relation $\prod_i \epsilon_i = +$:

$$\begin{aligned} \Omega_{1,2} &= (+,+,\pm,\pm); \\ \Omega_{3,4} &= (-,-,\pm,\pm); \\ \Omega_{5,6} &= (+,-,\pm,\mp); \\ \Omega_{7,8} &= (-,+,\pm,\mp). \end{aligned} \tag{5.20}$$

Model 5: cone over $\mathbf{PdP}_{4b}$

This theory admits only fixed points orientifold in which one gauge group (5) and three fields (X_{34}, X_{67}, X_{21}) are projected out; the unoriented quiver diagram is reported below

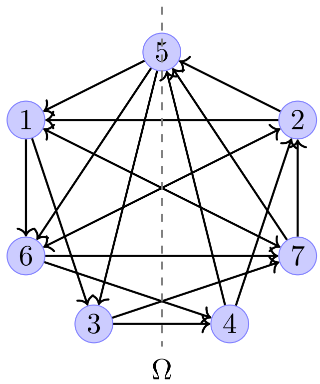

Figure 5.8. *Unoriented quiver diagram for fixed points orientifold of PdP_{4b}.*

Anomaly free conditions are

$$\begin{aligned} &2: N_5 + (N_2 + 4\epsilon_1) + N_7 - N_2 - N_4 - N_7 = 0; \\ &7: N_2 + N_5 + N_2 - N_7 - (N_7 + 4\epsilon_2) - N_4 = 0; \\ &4: N_2 + N_5 - N_7 - (N_4 + 4\epsilon_3) = 0; \end{aligned} \tag{5.21}$$

these equations lead to the relations

$$N_4 - N_5 = 4\epsilon_1,\ N_2 - N_7 = 4(\epsilon_1 + \epsilon_2),\ N_2 - N_7 = 2(\epsilon_1 + \epsilon_2),\ N_2 - N_7 = -4(\epsilon_3 - \epsilon_2); \tag{5.22}$$

that are compatible only if $\epsilon_1 = -\epsilon_2 = -\epsilon_3$. Moreover $\prod_i \epsilon_i = +$ and so the possible choices are

$$\begin{aligned} \Omega_1 &= (+,-,-,+); \\ \Omega_2 &= (-,+,+,-). \end{aligned} \tag{5.23}$$

5.2.1 Summary and future further development

In the previous few pages we briefly have studied, giving the possible non anomalous signs choices, orientifold of same reflexive polygon quiver theories. Models studied are couples that have toric diagrams that differ only for a blow up (or a blow down); these pair are:

- $\frac{\mathbb{C}^3}{\mathbb{Z}_3}$ and dP_1;
- $Y^{2,2}$ and PdP_2;
- PdP_{3a} and PdP_{4b}.

For the first two of them we note that the change in geometry, due to blow up, make geometrically not possible fixed line(s) orientifolds while does not touch the fixed points case. This is a relatively new general known feature [123]: fixed line(s) are allowed only for those models that are trapezoidal toric diagram. The important point here is that fixed line(s) orientifold are very related to geometry, so much so that the slightest change in the toric diagram can rule out its possibility. This, together with III scenario's examples, that are all at fixed points, seems to suggest that III scenario can occur only from fixed points orientifold. This is because III scenario change necessarily the geometry while fixed line(s) orientifolds are anchored to geometry.

To continue the study of the orientifolds of reflexive polygon quiver theories, the next step are models with more than one toric phase. The example of $\mathbb{F}_0$ reveals interesting behavior: Seiberg duality and orientifold commute. This is probably due to high degree of symmetry of $\mathbb{F}_0$ model, however would be interesting to study in general under what conditions this can happen. The difficult is that performing a Seiberg duality in an unoriented theory is not simple due to matter in symmetric or antisymmetric representations of the gauge group we want to dualize. Indeed, a general version of Seiberg duality with this kind of matter is, with the best knowledge of the author, not known. The way, almost not investigated at all, could be to use Kasteleyn matrix. In [89] authors explain how Seiberg duality act on Kasteleyn matrix; what is missing is the understanding of how orientifold act on this matrix. Then would be quite simple verify if Seiberg duality and orientifolds commute: if in both cases the final matrix is the same up to redefinitions of some kind, they commute.

5.3 Orientifolds action on toric diagrams from trivial and not trivial automorphisms

To perform the orientifold projection from toric data, we consider that the involution maps fields and groups in pairs. When a field is projected into a symmetric or antisymmetric representation of $SU(N)$, the contribution to the central charge is given by $\frac{N(N\pm1)}{2} \sim \frac{N^2}{2}$ at large N. In analogue way, an orthogonal or symplectic group contributes with gaugini in the adjoint representation; they, at large N, contribute with $\frac{N^2}{2}$ to the central charge. This means that the gaugini and matter

content is halved by the orientifold projection and so the central charge a^Ω from toric data is just half the parent one a. This argument reproduces quite trivially the I scenario constrain $a^\Omega = \frac{a}{2}$; however, as is evident, the third scenario can not be explained in these terms.
Probably the right formalism to understand how orientifolds act on toric diagrams is the one of automorphisms. Indeed, as explained well in [122], the orientifold acts as an automorphism on the homogeneous coordinates z_i, which are in one to one correspondence with the extremal point of the toric diagram and so with the trial charges a_i. This means that under a toric diagram automorphism the trial R-charges transoform in exactly the same way of the homogeneous coordinates. Hence a general toric diagram does not have a not trivial automorphism and it acts as an identity one

$$\sigma(z_i) = z_i \Rightarrow \sigma(a_i) = a_i. \tag{5.24}$$

This is the case of all the I scenario's examples; the central charge of the orientifold theory is simply calculated starting form the parent one by

$$a^\Omega = \frac{a(\sigma(a_1), ..., \sigma(a_d))}{2} = \frac{a(a_1, ..., a_d)}{2}, \tag{5.25}$$

where d is the number of trial charges. However, a particular toric diagrams can have not trivial automorphisms, which we denote as Σ. The possibility of these not trivial automorphisms open the doors to the III scenario in which the unoriented central charge is given in terms of the parent one as

$$a^\Omega = \frac{a(\Sigma(a_i)), ..., \Sigma(a_d))}{2}, \tag{5.26}$$

nevertheless, what is the form of these automorphisms to be able to describe III scenario's examples is, at the moment, not fully known. A partial answer is given by our proposal

$$\Sigma(z_i) = z_{i+1} \Rightarrow \Sigma(a_i) = a_{i+1} \tag{5.27}$$

as we see in a while with explicit examples, this does the job but only in some cases. For example automorphism 5.27 fails to explain the III scenario between PdP_{3b} and PdP_{3c}, and may be partially due to the difference in external points number between the two theories.

Example 1: SPP

We know from [121] that this theory belongs to the III scenario so we have to use, for example, the automorphism 5.27. Let us show how to construct the central charge a^Ω following this way.

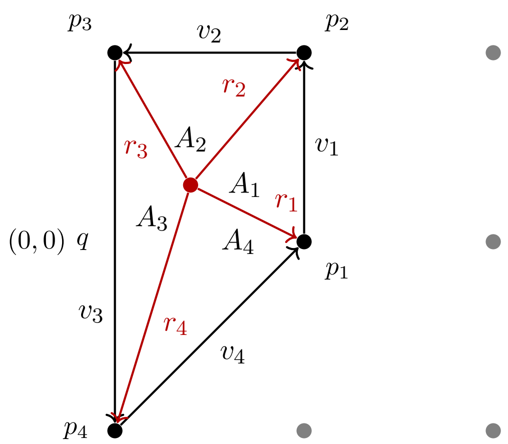

Figure 5.9. Toric diagram of SPP.

The parent central charge is

$$a = \frac{9}{32}\left[3 + (a_1 - 1)^3 + (a_2 - 1)^3 + (a_1 + a_2 - 1)^3 + 2\,(a_3 - 1)^3 + 2\,(a_4 - 1)^3\right] \tag{5.28}$$

and it is symmetric under $a_1 \leftrightarrow a_2$ and $a_3 \leftrightarrow a_4$; these constrains lead to the condition $y = \frac{x}{2}$ and, indeed, the central charge of the parent theory is maximized over only one parameter. Orientifolding the theory by applying 5.26 and 5.27 we get

$$a^\Omega = \frac{9}{64}\left[3 + (a_2 - 1)^3 + (a_3 - 1)^3 + (a_2 + a_3 - 1)^3 + 2\,(a_4 - 1)^3 + 2\,(a_1 - 1)^3\right] . \tag{5.29}$$

From perfect matchings and related fields of SPP,

$$\begin{aligned} p_1 &= \{X_{11}\ ,\ X_{12}\}; \\ p_2 &= \{X_{11}\ ,\ X_{21}\}; \\ p_3 &= \{X_{12}\ ,\ X_{20}\}; \\ p_4 &= \{X_{21}\ ,\ X_{02}\}; \\ q_1 &= \{X_{21}\ ,\ X_{31}\}; \\ q_2 &= \{X_{12}\ ,\ X_{13}\}, \end{aligned} \tag{5.30}$$

we can see that the orientifold maps fields as $p_1 \leftrightarrow p_2$ and $p_3 \leftrightarrow p_4$; this is the symmetry of the central charge of the parent theory. The guess seems to be that this symmetry is the $\mathbb{Z}_2$ one explicited by the orientifold action. Hence, we expect that $a_1 = a_2$ and $a_3 = a_4$ is still true for the orientifolded theory. Moreover, the central charge a^Ω is also symmetric under $a_2 \leftrightarrow a_3$ and this gives, together with the previous condition $y = \frac{x}{2}$, the point

$$\begin{aligned} \bar{x} &= \frac{1}{2} \\ \bar{y} &= \frac{1}{4}\ . \end{aligned} \tag{5.31}$$

At the superconformal fixed point 5.31 the central charge and the R-charges take the values

$$a^{\Omega} = \frac{81}{256} ,$$

$$a_1 = a_2 = a_3 = a_4 = \frac{1}{2} . \tag{5.32}$$

Some comments are in order now. First of all, the automorphism 5.27 reproduce the correct values of the central charge and R-charges; moreover, together with symmetries, it fixes all and nothing has to be maximized as we expected from [121]. We also note that the fields with R-charge $a_1 + a_2$, which is the adjoint field in the parent theory, has $R = 1$ and so it does not contribute to the orientifolded central charge.

Example 2: $\frac{\text{SPP}}{\mathbb{Z}_n'}$

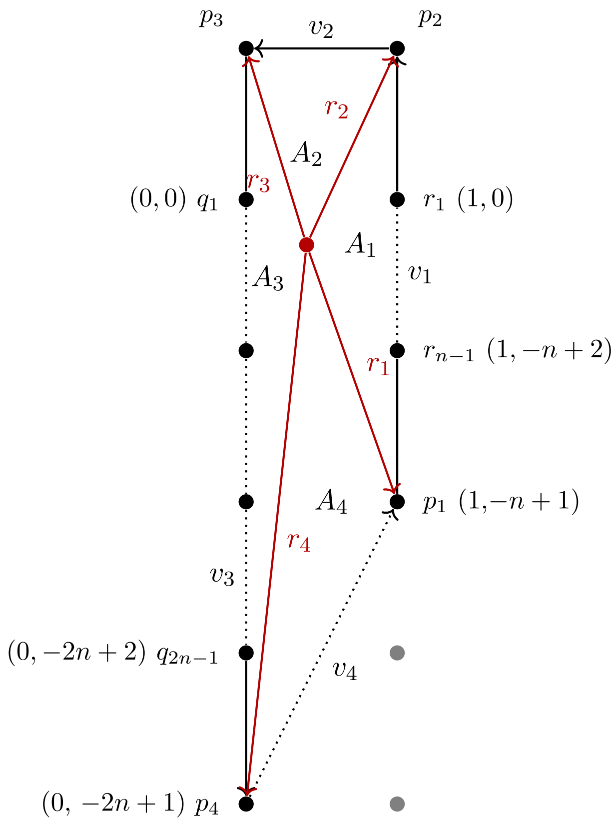

Figure 5.10. Toric diagram of non chiral orbifolds of SPP, SPP$/\mathbb{Z}_n'$.

Let us generalize what we have done in the previous example. As before, we want the orientifold preserve the symmetry of the parent theory that fix the loci of points $y = \frac{2-2n+nx}{2}$. Let us report all the interesting quantities to compute the parent central charge:

$$\begin{array}{lll} v_1 = (0, n)\ , & r_1 = (1 - x, -n + 1 - y)\ , & A_1 = \frac{n}{2}\,(1 - x)\,; \\ v_2 = (-1, 0)\ , & r_2 = (1 - x, 1 - y)\ , & A_2 = \frac{1}{2}\,(1 - y)\,; \\ v_3 = (0, -2n)\ , & r_3 = (-x, 1 - y)\ , & A_3 = \frac{1}{2}2nx \\ v_4 = (1, n)\ , & r_4 = (-x, -2n + 1 - y);\ , & A_4 = \frac{1}{2}\,(2n - 1 + y - nx)\,; \end{array} \tag{5.33}$$

set C and trial R-charges are given by

$$\begin{array}{lll} \langle v_4, v_1\rangle = n \to a_1, & a_1 = \frac{2x(1-y)}{n(2-x)} & = \frac{A_2A_3}{A_2A_3+A_3A_4+2A_4A_1+2A_1A_2} \\ \langle v_1, v_2\rangle = n \to a_2, & a_2 = \frac{2x(2n-1+y-x)}{n(2-x)} & = \frac{A_3A_4}{A_2A_3+A_3A_4+2A_4A_1+2A_1A_2} \\ \langle v_2, v_3\rangle = 2n \to a_3, & a_3 = \frac{2(1-x)(2n-1+y-x)}{n(2-x)} & = \frac{2A_4A_1}{A_2A_3+A_3A_4+2A_4A_1+2A_1A_2} \\ \langle v_3, v_4\rangle = 2n \to a_4, & a_4 = \frac{2(1-x)(1-y)}{n(2-x)} & = \frac{2A_1A_2}{A_2A_3+A_3A_4+2A_4A_1+2A_1A_2} \\ \langle v_4, v_2\rangle = 1 \to a_1 + a_2. & & \end{array} \tag{5.34}$$

The central charge $a^{\Omega}_{\frac{\text{SPP}}{\mathbb{Z}'_n}}$ can be easily extracted from the one of the parent theory using the automorphism 5.27:

$$a^{\Omega}_{\frac{\text{SPP}}{\mathbb{Z}'_n}} = \frac{9n}{64}\left[3 + (a_2 - 1)^3 + (a_3 - 1)^3 + (a_2 + a_3 - 1)^3 + 2\,(a_3 - 1)^3 + 2\,(a_1 - 1)^3\right], \tag{5.35}$$

as we expected, is equal to $n\,a^{\Omega}_{\text{SPP}}$.
Imposing $a_1 = a_2$, $a_3 = a_4$ from the symmetries of the parent theory and $a_2 = a_3$, $a_1 = a_4$ from the symmetries of $a^{\Omega}_{\frac{\text{SPP}}{\mathbb{Z}'_n}}$, we get the superconformal fixed point

$$\begin{aligned} \bar{x} &= \frac{1}{2}\ , \\ \bar{y} &= 1 - \frac{3n}{4}, \end{aligned} \tag{5.36}$$

and the values of the central charge and R-charges

$$\begin{aligned} a^{\Omega}_{\frac{\text{SPP}}{\mathbb{Z}'_n}} &= n\frac{81}{256}\ , \\ a_1 = a_2 = a_3 = a_4 &= \frac{1}{2}\ . \end{aligned} \tag{5.37}$$

These are the R-charges and the central charge of the I scenario orientifold of $L^{(\frac{3n}{2},\frac{3n}{2},\frac{3n}{2})}$ for n even and of the III scenario of $L^{(\frac{3n-1}{2},\frac{3n+1}{2},\frac{3n-1}{2})}$ for n odd; this is exactly what we expected from the explicit calculation in [121].

Counterexample 1: $\frac{\mathbb{C}^3}{\mathbb{Z}'_{3n}}$

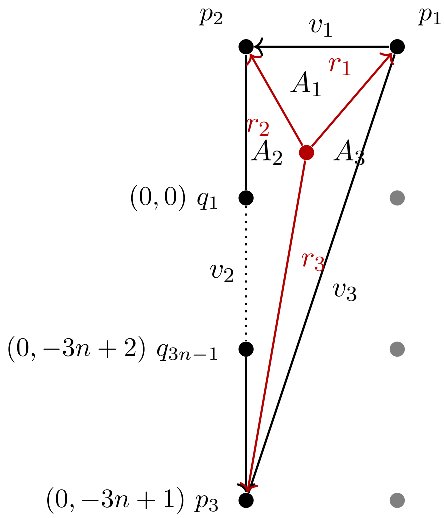

Figure 5.11. Toric diagram of the non chiral orbifolds of $\mathbb{C}^3$, $\frac{\mathbb{C}^3}{\mathbb{Z}'_{3n}}$.

This model has for n even two choices for orientifold projections: one in I scenario and the other in III scenario. Let us consider only the III scenario's orientifold.

$$\begin{aligned}
v_1 &= (-1,0)\ , & r_1 &= (1-x, 1-y)\ , & A_1 &= \tfrac{1}{2}\,(1-y) \\
v_2 &= (0,-3n)\ , & r_2 &= (-x, 1-y);\ , & A_2 &= \tfrac{1}{2}\,3nx \\
v_3 &= (1,3n)\ , & r_3 &= (-x,-3n+1-y);\ , & A_3 &= \tfrac{1}{2}(3n-1+y-3nx);
\end{aligned} \tag{5.38}$$

the set C and trial R-charges are given by

$$\begin{aligned}
\langle v_3, v_1\rangle &= 3n \to a_1\ , & a_1 &= 2x & &= \tfrac{2A_2}{A_1+A_2+A_3}; \\
\langle v_1, v_2\rangle &= 3n \to a_2\ , & a_2 &= \tfrac{2}{3n}\,(3n-1+y-3nx) & &= \tfrac{2A_3}{A_1+A_2+A_3}; \\
\langle v_2, v_3\rangle &= 3n \to a_3\ , & a_3 &= \tfrac{2}{3n}\,(1-y) & &= \tfrac{2A_1}{A_1+A_2+A_3}.
\end{aligned} \tag{5.39}$$

The parent central charge is given by

$$a_{\frac{\mathbb{C}^3}{\mathbb{Z}'_{3n}}} = \frac{27}{32}n\left[1+(a_1-1)^3+(a_2-1)^3+(a_3-1)^3\right] \tag{5.40}$$

and so using automorphism 5.27 we get

$$a^{\Omega}_{\frac{\mathbb{C}^3}{\mathbb{Z}'_{3n}}} = \frac{27}{64}n\left[1+(a_2-1)^3+(a_3-1)^3+(a_1-1)^3\right]\ ; \tag{5.41}$$

it is evident how in this case $\Sigma(z_i) = z_{i+1}$ no longer works since it leaves the same symmetry of the parent theory. We point out that automorphism 5.27 fails to describe also the III scenario' orientifold of PdP_{3c} and PdP_{3b}; however we can use reverse approach to get some clues: we know that in III scenario the adjoint fields should have $a_1 = 1$, which selects the point $\bar{x} = \frac{1}{2}$, while the bifundamental fields have $a_2 = a_3 = \frac{1}{2}$, which gives $\bar{x} = \frac{1}{2}$ and $\bar{y} = 1 - \frac{3n}{4}$. This seems to work as expected looking at [121], but we need to understand why. In particular, we need to find an automorphism such that, in the case of $\frac{\mathbb{C}^3}{\mathbb{Z}'_{3n}}$ model, the central charge $a^{\Omega}_{\frac{\mathbb{C}^3}{\mathbb{Z}'_{3n}}}$ incorporates the symmetry $a_2 + a_3 = a_1$.

Hence, automorphism 5.27 is not the final answer but, probably, only an example of an entire class of not trivial automorphism that are able to generate the III scenario.

Chapter 6

Conclusions

Let us give some conclusions, future perspectives and new ideas about this work.

In Section 4.4 we reinterpreted Butti-Zaffaroni's work. We wrote trial R-charges and the central charge in terms of triangle's areas and we were able to express these areas as work integral around their perimeter of the Reeb vector projected on the toric diagram. In the end, we can write the central charge as a function of these work integrals; this links intimately the central charge and the Reeb vector and so, in the AdS/CFT philosophy, a field theory quantity to a geometric one. The interesting point is that the work integral can be transformed in a volume integral and the Reeb vector has constant divergence. Hence, according to this naive analysis, the work integral can be reshaped as a volume. This could be useful to understand better, and maybe demonstrate in full generality, Gubser formula that links the central charge of a field theory to the volume of the Sasaki-Einstein space whose Calabi-Yau cone is the moduli space of the field theory. However, this naive approach can not be the all story since Calabi-Yau cones and their Sasaki-Einstein bases are trickier than spaces in which we can use this naive argument. Apart from this speculative idea, the interpretation and the meaning of this central charge rewriting remains a mystery. Furthermore, the structure equations' form, obtained in paragraph 4.4.3, seems to be too complicated to have, at the moment, applications that can lead to something interesting. However, if they remain pretty useless it is important to remember Steven Weinberg' words [124] and "to forgive yourself for wasting time" because "in the real world, it's very hard to know which problems are important, and you never know whether at a given moment in history a problem is solvable " and in the end "as you will never be sure which are the right problems to work on, most of the time that you spend in the laboratory or at your desk will be wasted."

In section 5.3 we have studied how the third scenario can come out, proposing a not trivial automorphism of the toric diagram that is able to describe some third scenario's examples. However, the proposed automorphism is not able to describe the unique chiral example of third scenario neither the non chiral class $\frac{\mathbb{C}^3}{\mathbb{Z}'_{3n}}$. Hence, it can not be the final answer and are needed new examples in order to better understand what is going on. Probably, the automorphism proposed is only an example of a entire class that can describe and explain third scenario. The main difficult is to

understand how and which automorphism can generate, in the unoriented theory, trial R-charges that are sum of some trial R-charges of the parent theory. Perhaps, what we would look for is some functional equation that, exploiting the constrains that we have to have for a third scenario, can give us a general form of the sought automorphism. This is; at the moment third scenario is quite obscure but what we know about it, is that probably only fixed points orientifold can generate it. Indeed, fixed line(s) orientifolds are too anchored to geometry and all the third scenario's examples are with fixed points.
It is important to underline that third scenario can have great potentiality and many applications. This is mainly due to the fact that before third scenario, the orientifolded moduli space could be only a projection of the parent theory moduli space while now the moduli space can change completely as in the case of PdP_{3c} and PdP_{3b}. This fact can open new unexplored consequences, for example in the context of moduli stabilization, that have to be studied in the future.

Acknowledgments

I would like to thank professor Fabio Riccioni and Salvo Mancani with whom I had the opportunity to collaborate for these seven months by immersing myself in the study of something completely new and recently proposed. Thanks to them I learned many new things but, above all, a way of approaching scientific research that I will carry with me in the years to come.

My biggest thanks is direct to Marilù without which I would never have taken the possibility to work on this thesis and I would have lost a great opportunity.

Appendix A

Differential operator realization of SUSY generators

Cosnsider a supertranslation (consisting in a SUSY transformation) on a superfield $Y(x,\theta,\bar{\theta})$

$$Y(x+\delta x,\theta+\delta\theta,\bar{\theta}+\delta\bar{\theta}) = e^{-i(\zeta Q+\bar{\zeta}\bar{Q})}Y(x,\theta,\bar{\theta})e^{i(\zeta Q+\bar{\zeta}\bar{Q})}, \tag{A.1}$$

where ζ^α and $\bar{\zeta}^{\dot{\alpha}}$ are spinorial parameters; the variation of the superfield under supertranslation is

$$\delta_{\zeta,\bar{\zeta}}Y(x,\theta,\bar{\theta}) \equiv Y(x+\delta x,\theta+\delta\theta,\bar{\theta}+\delta\bar{\theta}) - Y(x,\theta,\bar{\theta}). \tag{A.2}$$

Equation A.1 can be rewritten as

$$Y(x+\delta x,\theta+\delta\theta,\bar{\theta}+\delta\bar{\theta}) = e^{-i(\zeta Q+\bar{\zeta}\bar{Q})}e^{-i(xP+\theta Q+\bar{\theta}\bar{Q})}Y(0,0,0)e^{i(xP+\theta Q+\bar{\theta}\bar{Q})}e^{i(\zeta Q+\bar{\zeta}\bar{Q})}; \tag{A.3}$$

we focus on the last two exponentials and using the Baker-Campbell-Hausdorff formula we obtain schematically

$$e^{i(xP+\theta Q+\bar{\theta}\bar{Q})}e^{i(\zeta Q+\bar{\zeta}\bar{Q})} = e^{i(x+\theta\sigma\bar{\zeta}-\zeta\sigma\bar{\theta})P+i(\zeta+\theta)Q+i(\bar{\zeta}+\bar{\theta})\bar{Q}}, \tag{A.4}$$

if now we apply the same analysis to the first two exponentials we would obtain something very similar; this implies that[1]

$$\delta x^\mu = i[\theta^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\zeta}^{\dot{\beta}} - \zeta^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}],\ \delta\theta^\alpha = \zeta^\alpha,\ \delta\bar{\theta}^{\dot{\beta}} = \bar{\zeta}^{\dot{\beta}}. \tag{A.5}$$

Taylor expanding A.2 at first order and remembering A.5 we have

$$\delta_{\zeta,\bar{\zeta}}Y(x^\mu,\theta^\alpha,\bar{\theta}^{\dot{\beta}}) \simeq [\zeta^\alpha\partial_\alpha + \bar{\zeta}^{\dot{\beta}}\bar{\partial}_{\dot{\beta}} + i(\theta^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\zeta}^{\dot{\beta}} - \zeta^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}})\partial_\mu]Y(x^\mu,\theta^\alpha,\bar{\theta}^{\dot{\beta}}); \tag{A.6}$$

on the other hand, Taylor expanding at first order the same equation but using A.1 we get

$$\delta_{\zeta,\bar{\zeta}}Y(x^\mu,\theta^\alpha,\bar{\theta}^{\dot{\beta}}) \simeq -i\zeta^\alpha[Q_\alpha, Y(x^\mu,\theta^\alpha,\bar{\theta}^{\dot{\beta}})] + i\bar{\zeta}^{\dot{\beta}}[\bar{Q}_{\dot{\beta}}, Y(x^\mu,\theta^\alpha,\bar{\theta}^{\dot{\beta}})], \tag{A.7}$$

[1] Note that since we are combining two SUSY transformations we have an ordinary space-time translation, according to SUSY algebra.

compairing the last two equations we have

$$[Y(x^\mu,\theta^\alpha,\bar\theta^{\dot\beta}),Q_\alpha]=(-i\partial_\alpha-(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu)Y(x^\mu,\theta^\alpha,\bar\theta^{\dot\beta}):=\mathcal{Q}_\alpha Y(x^\mu,\theta^\alpha,\bar\theta^{\dot\beta});$$
$$[Y(x^\mu,\theta^\alpha,\bar\theta^{\dot\beta}),\bar Q_{\dot\beta}]=(+i\bar\partial_{\dot\beta}+\theta^\alpha(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu)Y(x^\mu,\theta^\alpha,\bar\theta^{\dot\beta}):=\bar{\mathcal{Q}}_{\dot\beta}Y(x^\mu,\theta^\alpha,\bar\theta^{\dot\beta});\tag{A.8}$$

where the calligraphic notation indicates the differential operator realizations. In conclusion we get equations 1.27.

Appendix B

Basics of conformal and superconformal algebra

Conformal algebra

In minkowsky space-time Poincaré algebra can be extended and we can define conformal trasformations: the most general locally causality preserving transformations. In general space-time conformal transformations are those transformations which leave the metric $g_{\mu\nu}$ invariant up to an arbitrary positive space-time dependent scale factor:

$$g_{\mu\nu}(x) \xrightarrow{CT} \Omega(x)^{-2} g_{\mu\nu}(x), \tag{B.1}$$

where CT stands for "Conformal Transformation". It is easy to see that conformal transformations change the length of an infinitesimal space-time interval but they leave angles invariant and preserve the causal structure. Let us now determine the conformal transformations in the case of a flat space-time metric; this will be of fundamental importance, for example, for our study of AdS/CFT correspondence. Consider an infinitesimal transformation

$$x^\mu \to x^\mu + \epsilon^\mu(x), \tag{B.2}$$

this will be a conformal transformation if the infinitesimal parameter $\epsilon^\mu(x)$ satisfies the so-called conformal Killing equation

$$(\eta_{\mu\nu}\partial_\rho\partial^\rho + (d-2)\partial_\mu\partial_\nu))\partial_\sigma\epsilon^\sigma(x) = 0 \tag{B.3}$$

where d is the dimension of the flat space-time. First note that we are forced to distinguish into two cases: $d = 2$ and $d \neq 2$; despite conformal symmetry in two dimensions is useful to study the worldsheet of a string or the Virasoro algebra, we no need this tool[1] for what we are interested in and so we consider only the case $d \neq 2$. Equation B.3 with the condition $d \neq 2$ is solved if $\epsilon^\mu(x)$ is at most of second order

$$\epsilon^\mu(x) = a^\mu + \omega^\mu_\nu x^\nu + \lambda x^\mu + b^\mu x^\nu x_\nu - 2(b^\nu x_\nu)x^\mu, \tag{B.4}$$

[1] It is interesting to point out that for $d = 2$ the conformal Killing equation reduces to the Cauchy-Riemann equations.

we already know that a^μ and ω^μ_ν are the parameters of translations and Lorentz transformations (LTs), but who are the others? Before answering this question it is important to note that the first two terms, as we know, leave the metric unchanged while the last three lead to a rescaling of the metric at each point: they are called dilatations and Special Conformal Trasformations (SCTs). The following table summarizes what we have said until now.

Name	$\epsilon^\mu(x)$	metric rescaling	generator
Translations	a^μ	no	P_μ
LTs	$\omega^{\mu\nu}x_\nu$	no	$M_{\mu\nu}$
Dilatations	λx^μ	yes, point independent	D
SCTs	$b^\mu x^\nu x_\nu - 2(b^\nu x_\nu)x^\mu$	yes, point dependent	K_μ

Table B.1. *Table summarizing the possible conformal transformations.*

The entire conformal algebra is

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho});\\
&[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu);\\
&[P_\mu, P_\nu] = 0;\\
&[K_\mu, K_\nu] = 0;\\
&[D, M_{\mu\nu}] = 0;\\
&[D, P_\mu] = iP_\mu;\\
&[D, K_\mu] = -iK_\mu;\\
&[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu);\\
&[K_\mu, P_\mu] = -2i(\eta_{\mu\nu}D - M_{\mu\nu}).
\end{aligned} \tag{B.5}$$

Note that obviously we have the Poincaré subalgebra $SO(d-1,1)$, moreover the generators of the conformal algebra can be grouped in such a way that the conformal algebra is the algebra $SO(d,2)$.

There is an important point to discuss: what the realization or not of conformal symmetry entails?. Firts of all, conformal invariance implies that the vacuum expectation value of the stress energy tensor's trace should vanish, however, theories interesting from the fenomenological point of view generally do not show conformal invariance at the quantum level and conformal symmetry is broken in an anomaly way by the introduction of a renormalization scale. Hence, non trivial contributions to the trace of the stress energy tensor involving the curvature emerge. In four dimensions the so-called trace anomaly is

$$\langle T^\mu_\mu \rangle = \frac{c}{16\pi^2} W^{\mu\nu\rho\sigma}W_{\mu\nu\rho\sigma} - \frac{a}{16\pi^2}(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2), \tag{B.6}$$

where $W^{\mu\nu\rho\sigma}$ is the Weyl tensor, $R^{\mu\nu\rho\sigma}$ is the Riemann tensor, $R^{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar. The term in parentheses is called Euler topological density because it is related, through Gauss-Bonnet theorem, to the Euler characteristic. Another consequence of conformal symmetry broken is that scaling dimensions of fields will be corrected by the anomalous dimensions $\gamma(g) = \mu\frac{d}{d\mu}\sqrt{ln(Z)}$ where g is the running coupling constant and Z are the wave function renormalization constants. Nevertheless, at particular regimes of RG flow, called fixed points of the beta function, theories show the full conformal invariance[2] at the quantum level. This is a good new since conformal invariance is so powerful that allow us to fix, up to a constant, the form of two and three points correlation function of the theory. Theories that show a conformal invariance are called Conformal Field Theories (CFTs)

As usual, the fields in a CFT transform in irreducible representations of the conformal algebra. Fields are classified according to their fixed scaling dimension Δ which contains their properties of deformation under a dilatation

$$\phi(x) \xrightarrow{dilatation} \lambda^{-\Delta}\phi(x), \tag{B.7}$$

hence fields are eigenstates[3] of the dilatation operator D. Moreover, in a CFT it is sufficient to consider particular fields, the so-called conformal primary, those fields that commute with the SCT operator. The reason why it is sufficient consider only primary fields is to note that, according to conformal algebra B.5, P_μ increase the scaling dimension while K_μ decreases it. Since in CFTs exist a lower bound for the scaling dimensions of fields follows that any conformal representation must contain operators of lowest dimension that have to be annihilated by K_μ: those are the primary fields and they have the lowest scaling dimension. Hence, all other fields, the so-called conformal descendants of the primary field, are obtained by acting with P_μ on the primary fields.

Superconformal algebra

We now study the consequences in case a supersymmetric theory is also conformal and so we talk about superconformal algebra. The generators of the superconformal algebra can be grouped into the generators of the conformal algebra, P_μ, $J_{\mu\nu}$, K_μ, D and the ordinary supercharges Q^I_α $\bar{Q}^{\dot\alpha}_I$. However, this is not the full set of generators, infact turn out that they do not close the algebra. Specifically the commutators between the supercharges and the SCT generator force us to introduce a new set of supercharges, S^I_α, $\bar{S}^{\dot\alpha}_I$, called conformal supercharges and in a number equal to the number of ordinary supercharges; the addition of these new supercharges closes the algebra. We not report the entire algebra since it is made up of about thirty commutators and anticommutations, but we refer to Appendix B of [42]. In the end, due to the presence of SCTs, supercharges are doubled and so, for example, a $d = 4$ and $\mathcal{N} = 4$ superconformal theory contains not sixteen supercharges but thirtytwo. As in the case of conformal algebra in which there are the primary fields, we consider the superconformal primary fields, which are those fields with the lowest scaling

[2] This is quite obvious since the beta function encodes how the theory changes when the energy scale changes, however if a theory is conformally invariant there cannot be a reference scale and so must be a vanishing beta function; this happens at a fixed point g^* of the RG flow, $\beta(g^*) = 0$.

[3] If the theory is not conformal invariant, scaling dimension is not fixed and under quantum corrections it undergoes a change. This change is called anomalous dimension. Hence in this case fields are not eigenstates of the dilatation operator.

dimansion within the superconformal multipltes. From the superconformal algebra turn out that, besides K_μ, also conformal supercharges lower the scaling dimension while ordinary supercharges and, as we already know, P_μ raise it[4]. Applying one of the last two operators on a superconformal primary field we can construct its descendants; a special kind of this descendants of the superconformal primary operator are the so-called superdescendants created applying the ordinary supercharges on a superconformal primary field. The interesting fact is that superdescendant operators are conformal primary operators and so each of them give rise to a conformal multiplet that is linked by SUSY transformation to all the others; this conformal multiplets are called Verma modules.

In superconformal algebra R-symmetry and so R-charges play an important role since they are related to the scaling dimension of the fields: given an operator $\mathcal{O}$ its R-charge must satisfy

$$\Delta_{\mathcal{O}} \geq \frac{3}{2}|R[\mathcal{O}]| \tag{B.8}$$

where the bound is saturated for the so-called chiral primaries fields which are ones that are annihilated by at least one ordinary supercharge.

The conformality of the theory makes the R-symmetry not anomalous. To see this consider a gauge theory $G_1 \times G_2 \times ... \times G_n$ with chiral superfields X_k in the representations r_k^i of the gauge factors G_i, since at a fixed point $\beta(g_i) = 0$ and $\Delta_k = 1 + \frac{1}{2}\gamma_{r_k^i} = \frac{3}{2}R_k \Rightarrow 3(R_k - 1) = \gamma_{r_k^i} - 1$ if we consider the NSVZ formula for the beta function we get

$$3T(adj_i) - \sum_{r_k^i} T(r_k^i)(1 - \gamma_{r_k^i}) = 3T(adj_i) + 3\sum_{r_k^i} T(r_k^i)(R_k - 1) = 0, \tag{B.9}$$

and this is exactly the anomaly cancellation for the $U(1)_R$ symmetry.

[4] K_μ lower by 1 while conformal supercharges lower by $\frac{1}{2}$. P_μ raise by 1 while ordinary supercharges raise by $\frac{1}{2}$.

Appendix C

Short review of complex geometry

First of all, let us define a complex manifold. Let be M a real $2m$ manifold and $\{U_i\}$ an open covering on M; on each open subset $\{U_i\}$, we define a coordinate chart to be the pair (U_i, ψ_i) where $\psi_i : U \to \mathbb{C}^m$ is a homeomorphism from U_i to an open subset of $\mathbb{C}^m$. The triad $(M, \{U_i, \psi_i\})$ is called complex manifold if for every non empty intersection $U_i \cap U_j$ the functions $\psi_{ij} = \psi_j \circ \psi_i^{-1}$, called transition functions, are holomorphic maps from $\mathbb{C}^m$ to itself; m is the complex dimension of the complex manifold. So a complex manifold of dimension m is a topological space that locally looks like $\mathbb{C}^m$.

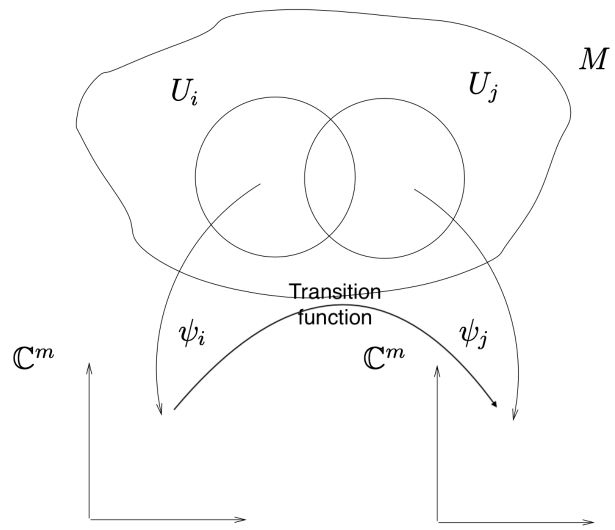

Figure C.1. *Schematic representation of a complex manifold of dimension m.*

In general a real manifold with dimension $m = 2k$ is not a complex manifold; to make sure that we need something more with respect to a general real manifold: a complex structure J. This is a smooth tensor field satisfying the relation[1] $J_a^b J_b^c = -\delta_a^c$,

[1]There is a little technicality: J must satisfies also $N_{bc}^a = J_b^d(\partial_d J_c^a - \partial_c J_a^d) - J_c^d(\partial_d J_a^b - \partial_b J_a^d) = 0$.

therefore J can be thought as a generalization of the well known imaginary unit i. Examples of complex manifolds are the complex projective spaces $\mathbb{CP}^m$; these are the spaces of complex lines through the origin of $\mathbb{C}^{m+1}$. This is constructed taking the space $\mathbb{C}^{m+1}\setminus\{0\}$ and quotient by the identification $(z_0, ..., z_m) \sim \lambda(z_0, ..., z_m)$. Other examples are the weighted projective spaces, they are constructed similar to the projective spaces but the identification in these cases is $(z_0, ..., z_m) \sim (\lambda^{i_0} z_0, ..., \lambda^{i_m} z_m)$ with $i_0, ..., i_m \in \mathbb{R}$.

Thanks to the complex structure relation $J^2 = -1$ turns out that the complexified tangent and cotanget bundles are decomposed:

$$\begin{aligned} T_{\mathbb{C}}M &= T^{(1,0)}M \oplus T^{(0,1)}M; \\ T^*_{\mathbb{C}}M &= T^{*(1,0)}M \oplus T^{*(0,1)}M; \end{aligned} \tag{C.1}$$

where $T^{(1,0)}M$ and $T^{(0,1)}M$ are the holomorphic and anti holomorphic tangent bundels while $T^{*(1,0)}M$ and $T^{*(0,1)}M$ are the holomorphic and anti holomorphic cotangent bundels. This decomposition is very useful since it can be applied at the k-th exterior power of the complexified cotangent bundle

$$\bigwedge^k T^*_{\mathbb{C}}M = \bigoplus_{j=0}^{k} \bigwedge^{j,k-j} M, \tag{C.2}$$

where we defined $\bigwedge^{p,q} M := \bigwedge^p T^{*(1,0)}M \otimes \bigwedge^q T^{*(0,1)}M$. Moreover also the exterior derivative admits a simple decomposition $d = \partial + \bar{\partial}$ where the operators ∂ and $\bar{\partial}$ are maps between (p, q)-form vector spaces $\Omega^{p,q}(M)$:

$$\begin{aligned} \partial &: \Omega^{p,q}(M) \to \Omega^{p+1,q}(M); \\ \bar{\partial} &: \Omega^{p,q}(M) \to \Omega^{p,q+1}(M). \end{aligned} \tag{C.3}$$

With the exterior derivative we can define[2] the cochain complex

$$0 \xrightarrow{\bar{\partial}_0} \Omega^{p,0}(M) \xrightarrow{\bar{\partial}_1} \Omega^{p,1}(M) \xrightarrow{\bar{\partial}_2} ... \xrightarrow{\bar{\partial}_{m-1}} \Omega^{p,m}(M) \xrightarrow{\bar{\partial}_m} 0 \tag{C.4}$$

with the property that $\partial_{n+1} \circ \partial_n = 0$[3] and so built up the Dolbeault cohomology groups as

$$H^{p,q}_{\bar{\partial}}(M) = \frac{Ker(\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{Im(\bar{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}. \tag{C.5}$$

As for de Rham cohomology groups $H^k_{dH}(M)$ that are defined in a similar way, Dolbeault cohomology groups measure how a closed form is not exact: $Ker(\bar{\partial})$ contains all the from that are mapped to zero by the action of $\bar{\partial}$, so they are closed forms, while $Im(\bar{\partial})$ contains all the forms that are mapped by $\bar{\partial}$, so they are exact forms. The dimensions of the Dolbeault cohomology groups are called Hodge numbers

$$h^{p,q} = dim(H^{p,q}_{\bar{\partial}}(M)); \tag{C.6}$$

[2] Obviously it is possible to retrace similar steps using ∂.

[3] This property underlines the fact that an exact form is also closed. Infact an exact form $\alpha \in \Omega^{p,n}(M)$ satisfies $\alpha = \bar{\partial}_n\beta$ for some $\beta \in \Omega^{p,n-1}(M)$ but now $(\partial_{n+1} \circ \partial_n)(\alpha) = 0$ for definition. Obviously is no true that a closed form is also exact.

turns out that they are not all independent.
Now that we have cohomology we can define the total Chern class:

$$c(E) = det\left(1 + \frac{i}{2\pi}F\right), \tag{C.7}$$

where E is a complex vector bundle over M and $F = dA + A \wedge A$ is the curvature form for the connection A; moreover total Chern class can be expanded as

$$c(E) = 1 + c_1(E) + ... + c_k(E) \tag{C.8}$$

where $c_i(E) \in H^{2i}_{dH}(M)$. Intuitively, total Chern class tell us how the complex vector bundle E is different form the trivial bundle.

We now define the so-called holomorphic line bundles. These are vector bundles (F, E, M, π) in which the projection map π is a holomorphic map, the fiber is $F = \mathbb{C}$ and for each $p \in M$ there exist an open neighborhood U of M and a biholomorphic map $\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}$ such that for each $u \in U$ the map ϕ_U takes the restriction on u of E to $\{u\} \times \mathbb{C}$ and this is a isomorphic between vector spaces.

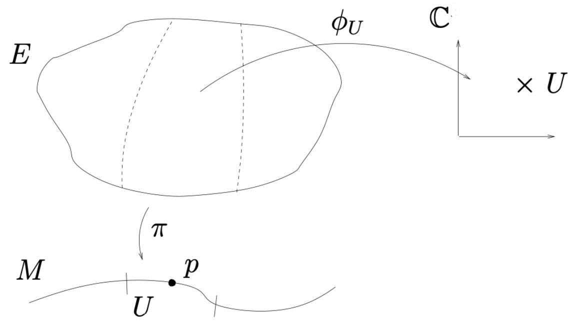

Figure C.2. *Schematic representation of line bundles.*

For us, the most important example of holomorphic line bundle is the so-called canonical bundle $K_M = \bigwedge^{m,0} T^{*(0,1)}M$; this is bundle on a complex manifold M of dimension m. Its sections[4] are the $(m, 0)$-forms, so they are forms with rank equal to the dimension of the manifold

Before talking a little of Kähler geometry we must introduce the concept of holonomy: this measures how vectors are transformed by parallel transport around a closed curve at a point $p \in M$. Strictly speaking, let be M a m-dimensional Riemannian manifold with metric g and affine connection[5] ∇, let be p a point in M and consider the set of closed path $\{l(t)|0 \leq t \leq 1, l(0) = l(1) = p\}$; take the vector field $X \in T_pM$ and parallel transport it along a closed path $l(t)$, we end up with a

[4]Recall that a section of a fiber bundle (F, E, B, π) is a continuous map $\sigma : B \to E$ such that, $\forall x \in B$, $\pi(\sigma(x)) = x$. Section are important tools since, for example in the case of tangent bundle (T_xM, TM, M, π) of a real manifold M with points x, they are vector fields.

[5]An affine connection is a geometric object on a smooth manifold which connects nearby tangent spaces. So it permits to compare vectors in different point of the manifold.

new vector field $Y \in T_pM$. Thus, the path $l(t)$ and the affine connection ∇ induce a linear transformation from T_pM to itself and the set of all these transformations is called holonomy group $Hol_p(M)$. Moreover, it is possible to show that the holonomy group is independent of the point. It is now obvious that $Hol(M)$ is contained in $GL(m)$.
Let us consider a complex manifold M with complex structure J that is endowed by a Riemannian hermitian metric g, so it satisfies $g_{ab} = J_a^c J_b^d g_{cd}$; we can define a symplectic 2-form ω called hermitian form by $\omega(v, w) := g(Jv, w)$ for all vector fields v, w on M. If ω is a closed form we call it Kähler form and g is dubbed Kähler metric. A complex manifold with Kähler metric is a Kähler manifold; it is possible to show, looking at the Kähler metric, that a Kähler manifold is that for which the parallel transport of a holomorphic vector remains a holomorphic vector. An important point on Kähler manifold is that is always possible to express the Kähler form ω in terms of a smooth function ϕ locally, this function is called Kähler potential and is not a case that has the same name of the Kähler potential of a SUSY theory: we have seen in the first chapter that a $\mathcal{N} = 1$ matter theory with a set of Φ and $\bar{\Phi}$ defines a Kähler metric by 1.41 where the scalar components of the chiral and anti chiral fields are interpeted as complex coordinates of a Kähler manifold.

Appendix D

III scenario between PdP$_{3c}$ and PdP$_{3b}$

This appendix is based on the work [120], figures are taken from there and for more details see directly the article. Let us report the toric diagrams of PdP$_{3c}$ and PdP$_{3b}$ and the unoriented quiver diagrams for the two theories.

Unoriented PdP$_{3b}$

We consider the orientifold projection Ω of PdP$_{3b}$ with two fixed lines and we choose the configuration with signs $\Omega = (-, +)$, as in figure below. We call the unoriented theory PdP$^{\Omega}_{3b}$. The gauge group $SU(N_5)$ is identified with $SU(N_3)$ while $SU(N_6)$ with $SU(N_2)$, moreover, $SU(N_1)$ becomes $Sp(N_1)$ and $SU(N_4)$ becomes $SO(N_4)$ since they lie on top of the fixed lines. The resulting theory has gauge group $Sp(N_1) \times SU(N_2) \times SU(N_3) \times SO(N_4)$, with the fields X_{35} and X_{62} belong to the antisymmetric and symmetric representations of the gauge groups $SU(N_3)$ and $SU(N_2)$ respectively.
Anomaly cancellation and β function vanishing conditions are satisfied only if $N_2 = N_3 = N_1 + 2 = N_4 - 2 = N$; then we find that the R-charges are

$$\begin{aligned} &R_{23} = 7 - 3\sqrt{5};\\ &R_{13} = R_{14} = R_{24} = 3 - \sqrt{5};\\ &R_{12} = R_{34} = R_{35} = R_{62} = 2\sqrt{5} - 4; \end{aligned} \tag{D.1}$$

up to $O\left(\frac{1}{N}\right)$ corrections. This imply that at large N, R-charges are the same of the parent theory PdP$_{3b}$ and the central charge can be computed to be

$$a^{\Omega} = \frac{27}{8} N^2 \left(5\sqrt{5} - 11\right), \tag{D.2}$$

exactly half of the parent central charge. This is the I scenario of PdP$_{3b}$.

Unoriented PdP$_{3c}$

Let us consider the orientifolds of PdP$_{3c}$. In figure below is shown that the dimer admits only the projections with fixed points, that project the group $SU(N_1)$, the

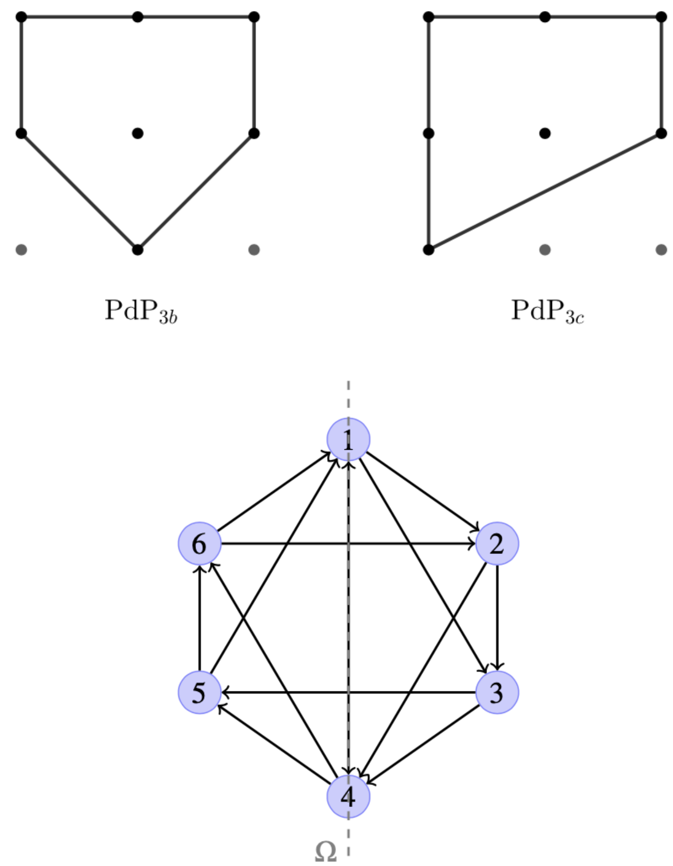

Figure D.1. *In the up panel are reported the toric diagrams for the models PdP_{3b} and PdP_{3c}. In the down panel is reported the unoriented quiver diagram for both models.*

group $SU(N_4)$, the field X_{35} and the field X_{62}. Since the parent theory has $N_W = 8$ we must have $\prod_i^4 \epsilon_i = +1$ and so only two inequivalent choices $\Omega_1 = (+,-,-,+)$ and $\Omega_2 = (-,+,-,+)$.

The case Ω_1 belongs to I scenario and so we look only to Ω_2. The unoriented theory obtained from $\Omega_2 = (-,+,-,+)$ has gauge groups $Sp(N1) \times SU(N2) \times SU(N3) \times SO(N4)$, and fields X_{35} and X_{62} are antisymmetric and symmetric representations of $SU(N_3)$ and $SU(N_2)$ respectively. The a maximization in this case is more subtle. If one took naively the limit $N \to \infty$ before solving the equation for vanishing β functions and $R[W] = 2$, one would obtain the R-charges and half the central charge of the parent theory. On the other hand, we find that the only consistent solution is $N_2 = N_3 = N_1 + 2 = N_4 - 2 = N$, for any finite value of N, exactly as in PdP_{3b} case. Hence, at leading order in $\frac{1}{N}$, the value of the R-charges turns out to be

$$\begin{aligned} &R_{23} = 7 - 3\sqrt{5}; \\ &R_{13} = R_{14} = R_{24} = 3 - \sqrt{5}; \\ &R_{12} = R_{34} = R_{35} = R_{62} = 2\sqrt{5} - 4; \end{aligned} \tag{D.3}$$

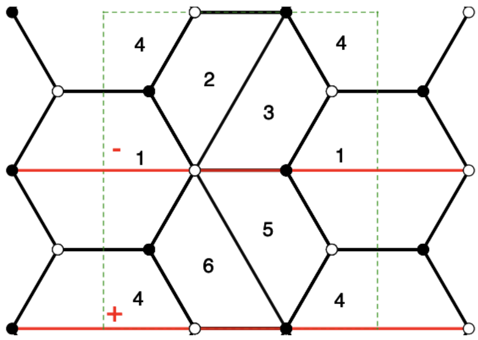

Figure D.2. *Dimer of PdP_{3b}, the dashed green line delimits the fundamental cell while the two red fixed lines and their signs represent the orientifold projection that yields the unoriented theory PdP^{Ω}_{3b}.*

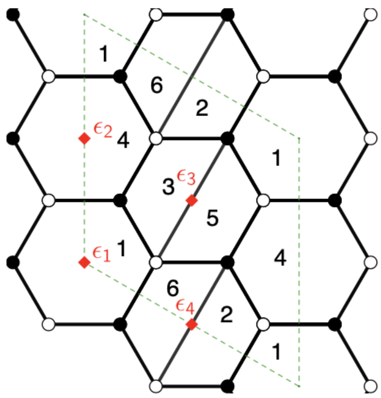

Figure D.3. *Dimer of PdP_{3c}, the dashed green line delimits the fundamental cell while the four red fixed points and their signs represent the orientifold projection that yields the unoriented theory $PdP^{\Omega_1}_{3c}$ or to $PdP^{\Omega_2}_{3c}$.*

which are different from the R-charges of the parent theory but are, surprisingly, equal to that of PdP^{Ω}_{3b}. The central charge is computed to be

$$a^{\Omega_2} = \frac{27}{8} N^2 (5\sqrt{5} - 11), \tag{D.4}$$

again equal to that of PdP^{Ω}_{3b}; moreover the ratio between the Ω_2-orientifolded central charge and the parent theory one is less that one half

$$\frac{a^{\Omega_2}}{a} = \frac{3\sqrt{3}}{2}(5\sqrt{5} - 11) \simeq 0,4685368. \tag{D.5}$$

The fact that a^{Ω_2} is less than halved with respect to the central charge of the parent theory can be taken as a sign of an RG flow towards the IR; and since R-charges and the central charge of $\text{PdP}^{\Omega_2}_{3c}$ are equal to those of PdP^{Ω}_{3b} suggest that the RG flow are going from $\text{PdP}^{\Omega_2}_{3c}$ in the UV to PdP^{Ω}_{3b} in the IR.

Appendix E

Acronyms

- ADS: Affleck-Dine-Seiberg;
- AdS: Anti de Sitter;
- BPS: Bogomol'nyi–Prasad–Sommerfield;
- BZ: Banks-Zaks;
- CFT: Conformal Field Theory;
- CTC: Closed Timelike Curve;
- CY: Calabi-Yau;
- DBC: Dirichlet boundary condition;
- dP: del Pezzo;
- eSQCD: electric Super Quantum ChromoDynamics;
- GLSM: Gauged Linear Sigma Model;
- GR: General Relativity;
- GSO: Gliozzi-Scherk-Olive;
- GUT: Great Unification Theory;
- IR: Infra-Red;
- KB: Kalb-Ramond;
- KK: Kaluza-Klein;
- lowSUGRA: low energy effective supergravity theory of suprestring theory;
- LT: Lorentz Transformations;
- mSQCD: magnetic Super Quantum ChromoDynamics;
- MSSM: Minimal Supersymmetric Standard Model;

- NBC: Neumann boundary condition;
- NG: Nambu-Goto;
- NS: Neveu-Schwarz;
- NSVZ: Novikov-Shifman-Vainshtein-Zakharov;
- PdP: Pseudo del Pezzo;
- QCD: Quantum ChromoDynamics;
- QFT: Quantum Field Theory;
- QG: Quantum Gravity;
- R: Ramond;
- RG: Renormalization Group;
- RNS:Ramond-Neveu-Schwarz;
- SUGRA: SuperGRAvity;
- SCRPC: Strongly Convex Rational Polyhedral Cone
- SCT: Special Conformal Transformation;
- SM: Standard Model;
- SQCD: Super Quantum ChromoDynamics;
- SUSY: SUperSYmmetry;
- SYM: Super Yang-Mills;
- SE: Sasaki-Einstein;
- UV: Ultra-Violet;
- VEV: Vacuum Expectation Value;
- WZIM: Wess-Zumino interacting model;

Bibliography

[1] Gervais, J.-L.; Sakita, B. (1971). "Field theory interpretation of supergauges in dual models". Nuclear Physics B. 34 (2)

[2] Yu. A. Golfand and E. P. Likhtman, «On the Extensions of the Algebra of the Generators of the Poincaré Group by the Bispinor Generators», in I. E. Tamm Memorial Volume Problems of Theoretical Physics, Eds. V. L. Ginzburg et al., (Nauka, Moscow 1972)

[3] D. V. Volkov, V. P. Akulov, Pisma Zh.Eksp.Teor.Fiz. 16 (1972) 621; Phys.Lett. B46 (1973) 109; V.P. Akulov, D.V. Volkov, Teor.Mat.Fiz. 18 (1974) 39

[4] H. Miyazawa (1966). "Baryon Number Changing Currents". Prog. Theor. Phys. 36 (6)

[5] H. Miyazawa (1968). "Spinor Currents and Symmetries of Baryons and Mesons". Phys. Rev. 170 (5)

[6] Georgi, H.; Glashow, S.L. (1974). "Unity of All Elementary Particle Forces". Physical Review Letters. 32 (8)

[7] GRAND UNIFIED THEORIES Revised October 2005 by S. Raby (Ohio State University).

[8] Supersymmetric Grand Unification, R. N. Mohapatra, arXiv:hep-ph/9801235

[9] Corbelli, E.; Salucci, P. (2000). "The extended rotation curve and the dark matter halo of M33". Monthly Notices of the Royal Astronomical Society. 311 (2)

[10] Dispense per il corso di Cosmologia Modulo A G.Tormen, Dipartimento di Astronomia, Universit‘a di Padova

[11] R. Slansky, "Group theory for Unified Model Building", Phys. Rep. 79 (1981)

[12] Lectures on Supersymmetry, Matteo Bertolini SISSA

[13] Modern Supersymmetry John Terning, Oxford Science Publications

[14] The quantum theory of fields. Vol.3:Supersymmetry, Cambridge, UK

[15] Coleman, Sidney; Mandula, Jeffrey (1967). "All Possible Symmetries of the S Matrix". Physical Review. 159 (5)

[16] Haag, Rudolf; Sohnius, Martin; Łopuszański, Jan T. (1975), "All possible generators of supersymmetries of the S-matrix", Nuclear Physics B, 88

[17] Introduction to Supersymmetry, Davide Cassani, INFN, Sezione di Padova, $https://userswww.pd.infn.it/\ cassani/susy_advanced_topics.pdf$

[18] J. Wess and J. Bagger. Supersymmetry and supergravity. Princeton University Press, Princeton, NJ, USA, 1992.

[19] Varieties of vacua in classical supersymmetric gauge theories Markus A. Luty (MIT), Washington Taylor IV (MIT). arXiv:hep-th/9506098

[20] Nima Arkani-Hamed and Hitoshi Murayama. Holomorphy, rescaling anomalies and exact beta functions in supersymmetric gauge theories. JHEP, 06:030, 2000.

[21] V.A. Novikov, Mikhail A. Shifman, A.I. Vainshtein, and Valentin I. Zakharov. Exact Gell-Mann-Low Function of Supersymmetric Yang-Mills Theories from Instanton Calculus. Nucl. Phys. B, 229:381–393, 1983.

[22] Notes on Seibergology EM-like duality SUSY and applications. Flip Tanedo Laboratory of Elementary Particle Physics, Cornell University, Ithaca, NY 14853, USA.

[23] Nick Dorey, Timothy J. Hollowood, Valentin V. Khoze, and Michael P. Mattis. The Calculus of many instantons. Phys. Rept., 371:231–459, 2002.

[24] Ian Affleck, Michael Dine, and Nathan Seiberg. Dynamical Supersymmetry Breaking in Supersymmetric QCD. Nucl. Phys. B, 241:493–534, 1984.

[25] Nathan Seiberg. Exact results on the space of vacua of four-dimensional SUSY gauge theories. arXiv:hep-th/9402044, 1994.

[26] T. Banks and A. Zaks, "On the Phase Structure of Vector-Like Gauge Theories with Massless Fermions," Nucl. Phys. B196 (1982) 189.

[27] N. Seiberg, "Electric - magnetic duality in supersymmetric nonAbelian gauge theories," Nucl. Phys. B435 (1995) 129–146, arXiv:hep-th/9411149.

[28] Edward Witten. An SU(2) Anomaly. Phys. Lett. B, 117:324–328, 1982.

[29] Hawking, S. W. (1974-03-01). "Black hole explosions?". Nature. 248 (5443): 30–31

[30] Hawking, S. (1975). Particle creation by black holes. Communications in Mathematical Physics, 43, 199–220.

[31] Black Holes and the Information Paradox Scientific American Vol. 276, No. 4 (APRIL 1997), pp. 52-57

[32] In Introduction to Quantum Effects in Gravity Mukhanov V., Winitzki S. (2007). Cambridge University Press.

[33] Black Holes: Thermodynamics, Information, and Firewalls, Robert B. Mann, SpringerBriefs in Physics.

[34] Abhay Ashtekar, "New variables for classical and quantum gravity," Phys. Rev. Lett., 57, 2244-2247, 1986.

[35] Abhay Ashtekar, "New Hamiltonian formulation of general relativity," Phys. Rev. D36, 1587-1602, 1987.

[36] Carlo Rovelli, Lee Smolin, "Discreteness of area and volume in quantum gravity" (1994): arXiv:gr-qc/9411005.

[37] The birth of string theory, Paolo Di Vecchia, arXiv:0704.0101.

[38] An introduction to string theory, Kevin Wray, $https : //math.berkeley.edu/\ kwray/papers/string_theory.pdf$

[39] Lectures on String Theory, David Tong, arXiv:0908.0333

[40] Superstring theory. M.B.Green, J.H.Schwarz, E.Witten. Cambridge University Press, Volume 1,2

[41] Teoria di stringa classica relativistica e sua applicazione nello spazio di de Sitter. Federico Manzoni, 2018. Archivio Sapienza.

[42] Gauge/Gravity Duality Foundations and Applications Martin Ammon, Johanna Erdmenger, Cambridge University Press 2015.

[43] F. Gliozzi, J. Scherk and D. I. Olive, "Supersymmetry, Supergravity Theories and the Dual Spinor Model", Nucl. Phys. B 122 (1977), 253.

[44] J. Polchinski. String theory. Vol. 1,2: An introduction to the bosonic string. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12 2007.

[45] Joseph Polchinski. Dirichlet Branes and Ramond-Ramond charges. Phys. Rev. Lett., 75:4724–4727, 1995.

[46] The Large N Limit of Superconformal Field Theories and Supergravity, Juan M. Maldacena, 1998, arXiv:hep-th/9711200.

[47] Susskind: The World as a Hologram, Journal of Mathematical Physics, Bd.36, 1995, S.6377 arXiv:hep-th/9409089

[48] Gerardus 't Hooft: Dimensional Reduction in Quantum Gravity, 1993 arXiv:gr-qc/9310026

[49] Merali, Zeeya (2011). "Collaborative physics: string theory finds a bench mate". Nature. 478 (7369): 302–304.

[50] Sachdev, Subir (2013). ”Strange and stringy”. Scientific American. 308(44): 44–51.

[51] Kovtun, P. K.; Son, Dam T.; Starinets, A. O. (2005). ”Viscosity in strongly interacting quantum field theories from black hole physics”. Physical Re-view Letters. 94 (11), arXiv:hep-th/040523.

[52] Introduction to the AdS/CFT correspondence, Alberto Zaffaroni, Universita‘ di Milano-Bicocca and INFN, sezione di Milano-Bicocca, Piazza della Scienza, 3; I-20126 Milano, Italy, https://virgilio.mib.infn.it/ zaffaron/lezioniLosannafin.pdf.

[53] Large N Field Theories, String Theory and Gravity Ofer Aharony, Steven S. Gubser, Juan Maldacena, Hirosi Ooguri, and Yaron Oz, 1999, arXiv:hep-th/9905111.

[54] TASI 2003 lectures on AdS/CFT Juan Maldacena, 2004 arXiv:hep-th/0309246.

[55] Quantum Fields in Curved Space, N. D. Birrell, Paul Davies, 1982.

[56] The conformal boundary of anti-de Sitter space-times, C. Frances, 2004, https://web.archive.org/web/20051013084907/http://mahery.math.u-psud.fr/ frances/ads-cft2.pdf.

[57] G. ’t Hooft (1974). "A planar diagram theory for strong interactions". Nuclear Physics B. 72 (3): 461.

[58] Toric Geometry, Sasaki-Einstein Manifolds and a New Infinite Class of AdS/CFT Duals, Dario Martelli, James Sparks, arXiv:hep-th/0411238.

[59] Lectures on complex geometry, Calabi-Yau manifolds and toric geometry, Vincent Bouchard, arXiv:hep-th/0702063.

[60] Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, Steven Weinberg, Wiley.

[61] An Example of a New Type of Cosmological Solutions of Einstein’s Field Equations of Gravitation, Rev. Mod. Phys. 21, 447-450 (1949) di Kurt Gödel

[62] Introduction to supergravity, Yoshiaki Tanii, Springer, Saitama, Japan.

[63] String Theory Dynamics In Various Dimensions, Edward Witten, arXiv:hep-th/9503124.

[64] "The space of Kähler metrics" E. Calabi Proceedings of the International Congess of Mathematicians vol 2. 1954

[65] "On Calabi’s conjecture and some new results in algebraic geometry" S.T. Yau Proceedings of the Nation Academy of Science of the U.S.A. 1977.

[66] Greene, Brian; Plesser, Ronen (1990). "Duality in Calabi–Yau moduli space". Nuclear Physics B. 338 (1): 15–37.

[67] Sasaki-Einstein Manifolds, James Sparks, arXiv:1004.2461.

[68] J. Cheeger, G. Tian, “On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay,” Invent. Math. 118 (1994), no. 3, 493-571.

[69] J. P. Gauntlett, D. Martelli, J. Sparks, D. Waldram, “Sasaki-Einstein metrics on S2 × S3,” Adv. Theor. Math. Phys. 8, 711 (2004), arXiv:hep-th/0403002.

[70] S. Smale, “On the structure of 5-manifolds,” Ann. Math. 75 (1962), 38-46.

[71] Th. Friedrich and I. Hath. "Einstein manifolds of dimension five with small first eigenvalue of the Dirac operator", J. Differential Geometry, 29 (1989) 263-279.

[72] M. Cvetic, H. Lu, D. N. Page, C. N. Pope, “New Einstein-Sasaki spaces in five and higher dimensions,” Phys. Rev. Lett. 95, 071101 (2005), arXiv:hep-th/0504225.

[73] D. Martelli, J. Sparks, “Toric Sasaki-Einstein metrics on S2 × S3,” Phys. Lett. B 621, 208 (2005), arXiv:hep-th/0505027.

[74] New Results in Sasaki-Einstein Geometry, James Sparks, 2008, arXiv:math/0701518.

[75] Superconformal Field Theory on Threebranes at a Calabi-Yau Singularity, 1998, Igor R. Klebanov, Edward Witten, arXiv:hep-th/9807080.

[76] P. Candelas and X. de la Ossa, “Comments on Conifolds,” Nucl. Phys. B342 (1990) 246.

[77] Steven S. Gubser. Einstein manifolds and conformal field theories. Phys. Rev. D, 59:025006, 1999.

[78] D. Anselmi, D.Z. Freedman, Marcus T. Grisaru, and A.A. Johansen. Nonperturbative formulas for central functions of supersymmetric gauge theories. Nucl. Phys. B, 526:543–571, 1998.

[79] Kenneth A. Intriligator and Brian Wecht. The Exact superconformal R symmetry maxi- mizes a. Nucl. Phys. B, 667:183–200, 2003.

[80] Toric geometry and local Calabi-Yau varieties: An introduction to toric geometry (for physicists), Cyril Closset, 2009, arXiv:0901.3695.

[81] String Dualities and Toric Geometry: An Introduction, Harald Skarke, 1997, arXiv:hep-th/9806059.

[82] What is a toric variety? David Cox, https://dacox.people.amherst.edu/lectures/tutorial.ps.

[83] introduction to toric varieties, William Fulton, Princeton University Press, 1993.

[84] B-Model Gromov-Witten Theory, Clader, Emily, Ruan, Yongbin, Birkhäuser, 2018.

[85] Brane Tilings and Quiver Gauge Theories, Rak-Kyeong Seong, 2013, Imperial College London Department of Physics.

[86] Reflexive polytopes: combinatorics and convex geometry, Benjamin Nill, https://personales.unican.es/santosf/anogia05/slides/Nill-anogia05.pdf.

[87] Bipartite Graphs, On-Shell Diagrams, and Bipartite Field Theories: a Computational Package in Mathematica, Daniele Galloni, 2017, arXiv:1702.03949.

[88] D-Brane Gauge Theories from Toric Singularities and Toric Duality, Bo Feng, Amihay Hanany, Yang-Hui He, 2000, arXiv:hep-th/0003085.

[89] Brane Dimers and Quiver Gauge Theories, Sebastian Franco, Amihay Hanany, Kristian D. Kennaway, David Vegh, Brian Wecht, 2005, arXiv:hep-th/0504110.

[90] Dimer models and toric diagrams, Amihay Hanany, Kristian D. Kennaway, 2005, arXiv:hep-th/0503149.

[91] A Comprehensive Survey of Brane Tilings, Sebastian Franco, Yang-Hui He, Chuang Sun, Yan Xiao, 2017, arXiv:1702.03958.

[92] Brane Tilings and Their Applications, Masahito Yamazaki, 2008, arXiv:0803.4474.

[93] The Homogeneous Coordinate Ring of a Toric Variety, 1993, arXiv:alg-geom/9210008.

[94] Thomas Delzant. Hamiltoniens périodiques et images convexes de l'application moment. Bulletin de la Société mathématique de France, 116(3):315–339, 1988.

[95] Eugene Lerman et al. Contact toric manifolds. Journal of Symplectic Geometry, 1(4):785– 828, 2002.

[96] Lectures on deformations of complex manifolds, Marco Manetti, 2005, arXiv:math/0507286.

[97] V. V. Batyrev and L. A. Borisov, Dual cones and mirror symmetry for generalized Calabi-Yau manifolds, . In *Greene, B. (ed.): Yau, S.T. (ed.): Mirror symmetry II* 71-86.

[98] V. V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, J. Alg. Geom. 3 (1994) 493–545.

[99] On the Classification of Reflexive Polyhedra, M. Kreuzer, H. Skarke, 1995, arXiv:hep-th/9512204.

[100] The number of vertices of a Fano polytope, C. Casagrande, 2004, arXiv:math/0411073.

[101] Volume and lattice points of reflexive simplices, Benjamin Nill, 2007, arXiv:math/0412480.

[102] Y. Imamura, "Anomaly cancellations in brane tilings," JHEP 06 (2006) 011.

[103] T. H. Buscher, "Path Integral Derivation of Quantum Duality in Nonlinear Sigma Models," Phys. Lett. B201 (1988) 466.

[104] Y. Imamura, "Anomaly cancellations in brane tilings," JHEP 06 (2006) 011.

[105] Y. Imamura, “Global symmetries and ’t Hooft anomalies in brane tilings,” JHEP 12 (2006) 041, hep-th/0609163.

[106] M. R. Douglas, B. R. Greene, and D. R. Morrison, Orbifold resolution by D-branes, Nucl.Phys. B506 (1997) 84–106, [hep-th/9704151].

[107] D. R. Morrison and M. R. Plesser, Nonspherical horizons. 1., Adv.Theor.Math.Phys. 3 (1999) 1–81, [hep-th/9810201].

[108] S. Franco and D. Vegh, Moduli spaces of gauge theories from dimer models: Proof of the correspondence, JHEP 0611 (2006) 054, [hep-th/0601063].

[109] Quivers, Tilings, Branes and Rhombi, Amihay Hanany, David Vegh, 2005, arXiv:hep-th/0511063.

[110] Unhiggsing the del Pezzo, Bo Feng, Sebastian Franco, Amihay Hanany, Yang-Hui He, 2002, arXiv:hep-th/0209228.

[111] Duality Walls, Duality Trees and Fractional Branes, Sebastian Franco, Amihay Hanany, Yang-Hui He, Pavlos Kazakopoulos, 2005, arXiv:hep-th/0306092.

[112] Corfu 05 lectures - part II: Orientifolds and model building, Emilian Dudas 2006 J. Phys.: Conf. Ser. 53 035.

[113] Dimers and Orientifolds, Sebastian Franco, Amihay Hanany, Daniel Krefl, Jaemo Park, Angel M. Uranga, David Vegh, 2007, arXiv:0707.0298.

[114] Augusto Sagnotti. Open Strings and their Symmetry Groups. In NATO Advanced Summer Institute on Nonperturbative Quantum Field Theory (Cargese Summer Institute), pages 0521–528, 9 1987.

[115] Massimo Bianchi and Augusto Sagnotti. On the systematics of open string theories. Phys. Lett., B247:517–524, 1990.

[116] Carlo Angelantonj and Augusto Sagnotti. Open strings. Phys. Rept., 371:1–150, 2002. [Erratum: Phys. Rept.376,no.6,407(2003)].

[117] A. Sagnotti, in: Cargese ’87, Non-Perturbative Quantum Field Theory, eds. G. Mack et al. (Pergamon Press, Oxford, 1988) p. 521.

[118] Massimo Bianchi, Gianluca Inverso, Jose Francisco Morales, and Daniel Ricci Pacifici. Unoriented Quivers with Flavour. JHEP, 01:128, 2014.

[119] Agostino Butti and Alberto Zaffaroni. R-charges from toric diagrams and the equivalence ofa-maximization and Z-minimization.JHEP, 11:019, 2005.

[120] A.Antinucci, S.Mancani and F.Riccioni, “Infrared duality in unoriented Pseudo del Pezzo,” Phys. Lett. B **811** (2020), 135902 doi:10.1016/j.physletb.2020.135902 [arXiv:2007.14749 [hep-th]].

[121] A.Antinucci, M.Bianchi, S.Mancani and F.Riccioni, “Suspended Fixed Points,” [arXiv:2105.06195 [hep-th]].

[122] Inaki-Garcia Etxebarria and Ben Heidenreich. S-duality in$N=1$ orientifold SCFTs.Fortsch.Phys., 65(3-4):1700013, 2017.

[123] Dimers, Orientifolds and Anomalies; Riccardo Argurio, Matteo Bertolini, Sebastián Franco, Eduardo García-Valdecasas, Shani Meynet, Antoine Pasternak, Valdo Tatitscheff; 2020, arXiv:2009.11291v2.

[124] Weinberg, S. Four golden lessons. Nature 426, 389 (2003). https://doi.org/10.1038/426389a

[125] Supergauge transformations in four dimensions, Wess Julius, Zumino Bruno, Nuclear Physics, Section B, Volume 70, Issue 1, p. 39-50, 1974.