Wick theorem

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1 Introduction

In Quantum Field Theory (QFT) the basic objects are the **fields**: already in the theory of newtonian gravitation or maxwellian electromagnetism we realized how the fundamental objects are fields. Gravitational and electromagnetic field in that cases. In full generality we can, assuming fields are distribution in the dual of the Schwartz space (so they must be in the space of tempered distribution) write fields using Fourier transform. The coefficients of this Fourier transform are the creation and annihilation operators associated to the field. If the field is a bosonic one (is a tensor representation of the Lorentz group) its creation and annihilation operator satisfy some commutation relations while if the field is a fermionic one (is a spinorial representation of the Lorentz group) its creation and annihilation operator satisfy some anticommutation relations. The simplest examples of bosonic and fermionic fields are the scalar one (describing for example the Higgs boson) and the Dirac one (describing for example the electron).

In the general framework of QFT, correlation functions, often referred to as correlators, are vacuum expectation values of time-ordered products of field operators. The **n-point correlation function** for an interacting QFT is defined as

$$G_n(x_1,\dots,x_n) := \langle\Omega|T[\phi_{i_1}(x_1)\dots\phi_{i_n}(x_n)]|\Omega\rangle; \tag{1}$$

where $|\Omega\rangle$ is the interacting vacuum and i_j with $j=1,...,n$ are multi-index specifying the Lorentz structure of the fields and the other quantum numbers. As in all the text we indicate fourvectors as $x^\mu := x$.

Correlation functions are the key object of study in QFT since they can be used to calculate various observables such as S-matrix elements, therefore probability transitions, thanks to the so-called LSZ formulae. Indeed, these formulae reduce the computation of scattering matrix element to the computation of a product of correlation functions of fields where the vacuum is the interacting one. However, this is not a very useful result by itself; we need to know how the rewrite correlation functions on the interacting vacuum as correlation functions on the free theory vacuum. This is possible using the **interaction picture** and a single correlation function on the interacting vacuum is rewrite as an infinite sum of correlation function on the free theory vacuum

$$G_n(x_1,\dots,x_n) = \frac{\langle 0|T[\phi^{(I)}_{i_1}(x_1)\dots\phi^{(I)}_{i_n}(x_n)e^{iS_{int}[\phi^{(I)}]}]|0\rangle}{\langle 0|e^{iS_{int}[\phi^{(I)}]}|0\rangle}, \tag{2}$$

where the apex "(I)" underline these are fields in the interaction picture while the subscript "int" meas that that is the interaction term in the action of the theory. The infinite sum is given by

Taylor expanding the Boltzmann-like weight due to the action of the model. Truncate the Taylor expansion, from the physical point of view, is called a **perturbative expansion**. The Taylor expansion truncation is possible if and only if there is a quantity we can think goes to zero or that it is small. In the case of QFTs this quantity is the **coupling constant** λ, so the parameter that say how strong is the interaction. On the one hand if the interaction is weak then $\lambda << 1$, we expect that the interaction does not change things so much and a perturbative expansion is possible; on the other hand, if the interaction is strong then $\lambda >> 1$, we expect that the interaction change a lot of things and a perturbative expansion is lost forever. For example the Quantum ElectroDynamis (QED) is perturbative while the Quantum ChromoDynamis (QCD) is perturbative only at high energy while became non-perturbative at low energy.
A toy model example we can present to get familiar with (1) is provided by the the so-called scalar $\lambda\phi^4$ model. The interaction hamiltonian density is given by

$$\mathcal{H}_{int}^{(I)} = \frac{\lambda}{4!}[\phi^{(I)}(x)]^4; \tag{3}$$

so the action describing the interaction is given by

$$S_{int}[\phi^{(I)}(x)] = e^{i\int d^4x \frac{\lambda}{4!}[\phi^{(I)}(x)]^4} = \sum_k \frac{(-i\int d^4x \frac{\lambda}{4!}[\phi^{(I)}(x)]^4)^k}{k!}; \tag{4}$$

so (2) became

$$G_n(x_1,\ldots,x_n) = \frac{\langle 0|T[\phi_{i_1}^{(I)}(x_1)\ldots\phi_{i_n}^{(I)}(x_n)\sum_k \frac{(i\int d^4x \frac{\lambda}{4!}[\phi^{(I)}(x)]^4)^k}{k!}]|0\rangle}{\langle 0|\sum_k \frac{(i\int d^4x \frac{\lambda}{4!}[\phi^{(I)}(x)]^4)^k}{k!}|0\rangle}. \tag{5}$$

This infinite series can be truncate if and only if $\lambda << 1$ and how much we can truncate depends on how much λ is less than one. The term $k = 0$ is called **free theory or 0th order** and the correlation function on the interacting vacuum are the same of those on the free theory vacuum simply because there is no interaction if $\lambda = 0$. The term $k = 1$ is called the **tree level or 1th order**, here there is nothing of quantum but just relativistic. Quantum effects arise from terms with $k \geq 2$, these terms are called **quantum corrections** however some classical effects can be generated by quantum corrections as well. This separation of the terms depend also on the theory, for example in QED the tree level is given by the term $k = 2$ simply because term with $k = 1$ does not respect conservation of energy.

The **Wick theorem** enter in the game at this point: although the computation of correlation functions on the free theory vacuum is obviously simpler than that on the interacting theory vacuum, having to compute correlation functions with higher and higher points is not a simple task. Wick theorem allows us to reduce multi-point correlation functions to sums of products of two-point correlation functions. As we know the two-point correlation function is nothing but the propagator of the field. Every multi-point correlation function on the interacting vacuum can be represented by a set of graphical pictures with lines and dots called **Feynman diagrams**. The internal lines represent the propagators, the external one represent the fields string $\{\phi_{i_1}^{(I)}(x_1),...,\phi_{i_n}^{(I)}(x_n)\}$ while the dots represent the interacting vertexes inside the action functional $S_{int}[\phi_{i_1}^{(I)}]$. Diagram for the free theory are only lines since the theory is not coupled, tree level diagrams do not contain loops while quantum correction diagrams contain loops. So loops are the quantum corrections, even if in same case they can lead classical effects.

The separation of terms in free theory, tree level and quantum corrections seems quite mysterious but it is an easy consequence of restoring the Planck constant: the action will carry with it a factor of $\hbar^{-1}$ so the also the part of the action describing the interaction will carry a factor of $\hbar^{-1}$, on the other hand the propagator, that is the inverse of the kinetic term, will carry a factor $\hbar^{+1}$. From this we see that for a diagram with A vertices and B internal lines the number of independent momenta is $C = B - A + 1$ and corresponds to the number of loops. Associating a factor of $\hbar^{-1}$ for the A vertices and $\hbar^{+1}$ for the B propagators yields an overall factor

$$\hbar^{B-A+1} = \hbar^{C}. \tag{6}$$

So if we have zero loops we have no powers of $\hbar$ and so the tree level. However since the mass carries factor of $\hbar^{-1}$ some extra powers of $\hbar$ enter in the propagator according to how the mass enter in the propagator (so strongly depend on the type of field) and therefore also the quantum correction terms can give rise to classical effects due to cancellation of $\hbar$ powers.

2 The normal ordering

We first need to introduce the normal ordering of creation and annihilation operators.

*Definition (**normal ordering of creation and annihilation operators**): we say that a product of creation and annihilation operators is normal ordered if all creation operators are to the left of all annihilation operators. We indicate the normal order as* $: \cdot :$.

The process of normal ordering is particularly important for a quantum mechanical hamiltonian system. When quantizing a classical hamiltonian there is some freedom when choosing the operator order and these choices lead to differences in the ground state energy. This is called the problem of ordering in Quantum Mechanics.

Example 1: $N = 1$ boson

Let us consider $N = 1$ boson field with creation and annihilation operators $\hat{b}^\dagger$ and $\hat{b}$ satisfying the commutation relations

$$\left[\hat{b}^\dagger, \hat{b}^\dagger\right] = 0; \quad \left[\hat{b}, \hat{b}\right] = 0; \quad \left[\hat{b}, \hat{b}^\dagger\right] = 1. \tag{7}$$

We have

$$\begin{aligned} :\hat{b}^\dagger \, \hat{b}: &= \hat{b}^\dagger \, \hat{b}; \\ :\hat{b} \, \hat{b}^\dagger: &= \hat{b}^\dagger \, \hat{b}; \end{aligned} \tag{8}$$

note that essentially commutation relations do not matter, we simply we move the creation operator to the left of the annihilation operator. It is important to note that normal orderind is not a linear map since

$$\hat{b}^\dagger \hat{b} = :\hat{b}\hat{b}^\dagger: \overset{(7)}{=} :1 + \hat{b}^\dagger \hat{b}: \overset{?}{=} :1: + :\hat{b}^\dagger \hat{b}: = 1 + \hat{b}^\dagger \hat{b} \neq \hat{b}^\dagger \hat{b}. \tag{9}$$

Example 2: $N > 1$ **bosons**

Let us consider $N > 1$ boson fields with creation and annihilation operators $\hat{b}_i^\dagger$ and $\hat{b}_i$ satisfying the commutation relations

$$\left[\hat{b}_i^\dagger, \hat{b}_j^\dagger\right] = 0; \quad \left[\hat{b}_i, \hat{b}_j\right] = 0; \quad \left[\hat{b}_i, \hat{b}_j^\dagger\right] = \delta_{ij}. \tag{10}$$

For example for $N = 3$ bosons we have

$$\begin{aligned} &: \hat{b}_i^\dagger \, \hat{b}_j \, \hat{b}_k := : \hat{b}_j \, \hat{b}_i^\dagger \, \hat{b}_k := : \hat{b}_k \hat{b}_j \, \hat{b}_i^\dagger := \hat{b}_i^\dagger \, \hat{b}_j \, \hat{b}_k \stackrel{(7)}{=} \hat{b}_i^\dagger \, \hat{b}_k \, \hat{b}_j; \\ &: \hat{b}_i^\dagger \, \hat{b}_j^\dagger \, \hat{b}_k := : \hat{b}_j^\dagger \, \hat{b}_i^\dagger \, \hat{b}_k := : \hat{b}_k \hat{b}_j^\dagger \, \hat{b}_i^\dagger := \hat{b}_i^\dagger \, \hat{b}_j^\dagger \, \hat{b}_k \stackrel{(7)}{=} \hat{b}_j^\dagger \, \hat{b}_i^\dagger \, \hat{b}_k. \end{aligned} \tag{11}$$

Example 3: $N = 1$ **fermion**

Let us consider $N = 1$ fermion field with creation and annihilation operators $\hat{f}^\dagger$ and $\hat{f}$ satisfying the commutation relations

$$\left\{\hat{f}^\dagger, \hat{f}^\dagger\right\} = 0; \quad \left\{\hat{f}, \hat{f}\right\} = 0; \quad \left\{\hat{f}, \hat{f}^\dagger\right\} = 1. \tag{12}$$

We have

$$\begin{aligned} &: \hat{f}^\dagger \, \hat{f} : = \hat{f}^\dagger \, \hat{f}; \\ &: \hat{f} \, \hat{f}^\dagger : = -\hat{f}^\dagger \, \hat{f}; \end{aligned} \tag{13}$$

note that also in this case anticommutation relations do not matter, we simply we move the creation operator to the left of the annihilation operator: the minus sign is not due to anticommutators but to the fact that these operators are fermionic one (the are anticommuting objects whether or not they are quantum operators quantized using anticommutators).

The vacuum expectation value of a normal ordered product of creation and annihilation operators is zero. This is because by definition the creation and annihilation operators satisfy

$$\langle 0 | \hat{a}_i^\dagger = 0, \quad \hat{a}_i | 0 \rangle = 0 \quad i = 1, ..., N. \tag{14}$$

So, in full generality, even if an operator $\hat{O}$ is such that $\langle 0 | \hat{O} | 0 \rangle \neq 0$ its normal ordering satisfy $\langle 0 | : \hat{O} : | 0 \rangle = 0$ and this is of fundamental importance for the Wick theorem application to QFT.

3 The Wick theorem

Before introducing Wick theorem we need to define the contraction of creation and annihilation operators.

*Definition (**contraction of creation and annihilation operators**): given two creation and/or annihilation operators $\hat{A}$ and $\hat{B}$ we define their contraction as* $\overset{\sqcap}{\hat{A}\hat{B}} := \hat{A}\,\hat{B} - :\hat{A}\,\hat{B}:$.

We consider a set $\{\hat{a}_i, \hat{a}_i^\dagger\}_{i=1}^N$ of creation and annihilation operators that satisfy $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ for bosons and $\{\hat{a}_i, \hat{a}_j^\dagger\} = \delta_{ij}$ for fermions. We have

$$\begin{aligned}
&\wick{\c{\hat{a}}_i \c{\hat{a}}_j} = \hat{a}_i\, \hat{a}_j\, - :\hat{a}_i\, \hat{a}_j: \, = 0 = \wick{\c{\hat{a}}_i^\dagger \c{\hat{a}}_j^\dagger} = \hat{a}_i^\dagger\, \hat{a}_j^\dagger\, - :\hat{a}_i^\dagger\, \hat{a}_j^\dagger: \, = 0 = \wick{\c{\hat{a}}_i^\dagger \c{\hat{a}}_j} = \hat{a}_i^\dagger\, \hat{a}_j\, - :\hat{a}_i^\dagger\, \hat{a}_j: \, = 0; \\
&\wick{\c{\hat{a}}_i \c{\hat{a}}_j^\dagger} = \hat{a}_i\, \hat{a}_j^\dagger\, - :\hat{a}_i\, \hat{a}_j^\dagger: \, = \delta_{ij};
\end{aligned} \tag{15}$$

These relations are true for both bosonic or fermionic operators operators because of the way normal ordering is defined. Let us give some examples.

Example 1

$$\hat{a}_i\, \hat{a}_j^\dagger\, \hat{a}_k = (\pm \hat{a}_j^\dagger\, \hat{a}_i + \delta_{ij})\hat{a}_k = \pm \hat{a}_j^\dagger\, \hat{a}_i\, \hat{a}_k + \delta_{ij}\hat{a}_k = \pm \hat{a}_j^\dagger\, \hat{a}_i\, \hat{a}_k + \wick{\c{\hat{a}}_i \c{\hat{a}}_j^\dagger}\hat{a}_k = \, :\hat{a}_i\, \hat{a}_j^\dagger \hat{a}_k: + :\wick{\c{\hat{a}}_i \c{\hat{a}}_j^\dagger}\, \hat{a}_k: \tag{16}$$

Example 2

$$\begin{aligned}
\hat{a}_i\, \hat{a}_j^\dagger\, \hat{a}_k\, \hat{a}_l^\dagger &= (\pm \hat{a}_j^\dagger\, \hat{a}_i + \delta_{ij})(\pm \hat{a}_l^\dagger\, \hat{a}_k + \delta_{kl}) = \\
&= \hat{a}_j^\dagger\, \hat{a}_i\, \hat{a}_l^\dagger\, \hat{a}_k \pm \delta_{kl}\hat{a}_j^\dagger\, \hat{a}_i \pm \delta_{ij}\hat{a}_l^\dagger \hat{a}_k + \delta_{ij}\delta_{kl} = \\
&= \hat{a}_j^\dagger(\pm \hat{a}_l^\dagger\, \hat{a}_i + \delta_{il})\hat{a}_k \pm \delta_{kl}\hat{a}_j^\dagger\, \hat{a}_i \pm \delta_{ij}\hat{a}_l^\dagger \hat{a}_k + \delta_{ij}\delta_{kl} = \\
&= \pm \hat{a}_j^\dagger \hat{a}_l^\dagger\, \hat{a}_i \hat{a}_k + \delta_{il}\hat{a}_j^\dagger\, \hat{a}_k \pm \delta_{kl}\hat{a}_j^\dagger\, \hat{a}_i \pm \delta_{ij}\hat{a}_l^\dagger \hat{a}_k + \delta_{ij}\delta_{kl} = \\
&= \, :\hat{a}_i\, \hat{a}_j^\dagger\, \hat{a}_k\, \hat{a}_l^\dagger: + :\wick{\c{\hat{a}}_i \hat{a}_j^\dagger \hat{a}_k \c{\hat{a}}_l^\dagger}: + :\hat{a}_i\, \hat{a}_j^\dagger\, \wick{\c{\hat{a}}_k \c{\hat{a}}_l^\dagger}: + :\wick{\c{\hat{a}}_i \c{\hat{a}}_j^\dagger}\, \hat{a}_k\, \hat{a}_l^\dagger: + \, :\wick{\c{\hat{a}}_i \c{\hat{a}}_j^\dagger}\, \wick{\c{\hat{a}}_k \c{\hat{a}}_l^\dagger}:.
\end{aligned} \tag{17}$$

Note that in both examples we have placed contractions inside normal ordering, however as contractions are not creation or annihilation operators they can be brought in or out without any problem.

For more general product using commutators or anticommutators can be very combersome. With the goal to simplify this procedure, Wick theorem enters in the game.

*Theorem (**Wick**): Given N bosonic and/or fermionic creation and/or annihilation operators, the product of this N creation and/or annihilation operators can be rewritten as a sum of normal* *ordered terms containing contractions and the following formula holds*

$$
\begin{aligned}
\underbrace{\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\dots}_{N} = {} & \underbrace{:\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\dots:}_{N} \pm \\
& \pm \sum_{singles} \underbrace{:\overline{\hat{A}\hat{B}}\hat{C}\hat{D}\hat{E}\hat{F}\dots:}_{N} \pm \\
& \pm \sum_{doubles} \underbrace{:\overline{\hat{A}\hat{B}}\,\overline{\hat{C}\hat{D}}\hat{E}\hat{F}\dots:}_{N} \pm \\
& \pm \sum_{triples} \underbrace{:\overline{\hat{A}\hat{B}}\,\overline{\hat{C}\hat{D}}\,\overline{\hat{E}\hat{F}}\dots:}_{N} \pm \\
& \pm \dots \pm \\
& \pm \; full\ contracted\ term.
\end{aligned}
\tag{18}
$$

Minus signs are introduced whenever the order of two fermionic operators is swapped to ensure the contracted terms are adjacent in product.

*Proof (**Wick theorem**).*
We use induction to prove the theorem for bosonic creation and annihilation operators. The $N = 2$ base case is trivial, because there is only one possible contraction

$$
\hat{A}\hat{B} = :\hat{A}\hat{B}: + (\hat{A}\,\hat{B}\; - :\hat{A}\,\hat{B}:) = :\hat{A}\hat{B}: + \overline{\hat{A}\hat{B}}. \tag{19}
$$

In general, the only non-zero contractions are between an annihilation operator on the left and a creation operator on the right. Suppose that Wick's theorem is true for $N-1$ operators $\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\dots$, and consider the effect of adding an Nth operator $\hat{A}$ to the left of $\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\dots$ to form $\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\dots$. By Wick's theorem applied to $N-1$ operators, we have

$$
\begin{aligned}
\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\dots = {} & \hat{A}{:}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\dots{:}+ \\
& + \hat{A} \sum_{\text{singles}} :\overline{\hat{B}\hat{C}}\hat{D}\hat{E}\hat{F}\dots:+ \\
& + \hat{A} \sum_{\text{doubles}} :\overline{\hat{B}\hat{C}}\,\overline{\hat{D}\hat{E}}\hat{F}\dots:+ \\
& + \hat{A}\dots + \\
& + \hat{A}\; full\ contracted\ term.
\end{aligned}
\tag{20}
$$

$\hat{A}$ is either a creation operator or an annihilation operator. On the one hand, if $\hat{A}$ is a creation operator, all above products, such as $\hat{A}{:}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\dots{:}$, are already normal ordered and require no further manipulation. Because $\hat{A}$ is to the left of all annihilation operators in $\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\dots$, any contraction involving it will be zero. Thus, we can add all contractions involving $\hat{A}$ to the sums without changing their value. Therefore, if $\hat{A}$ is a creation operator, Wick's theorem holds for $\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\ldots$. On the other hand, if $\hat{A}$ is an annihilation operator, in order to move $\hat{A}$ from the left-hand side to the right-hand side of all the products, we repeatedly swap $\hat{A}$ with the operator immediately right of it (let us call it $\hat{X}$). Every time we make a swap we apply $\hat{A}\hat{X} = :\hat{A}\hat{X}: + \overbrace{\hat{A}\hat{X}}$ from (19). Once we do this, all terms will be normal ordered and all terms added to the sums by pushing $\hat{A}$ through the products correspond to additional contractions involving $\hat{A}$ that reconstruct Wick theorem also for $\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\ldots$. By introducing the appropriate minus signs, the proof can be extended to fermionic creation and annihilation operators. $\square$

We have given proof for Wick's theorem in the case of creation and annihilation operators, however we are interested in quantum fields. The key point is that quantum fields are decomposed into sums of creation and annihilation operators, so we see that the theorem must also hold in the case of quantum fields; this is because we can break each field up into its creation and annihilation parts

$$\phi_i^{(I)}(x) = \phi_i^{(I+)}(x) + \phi_i^{(I-)}(x). \tag{21}$$

where $\phi_i^{(I+)}(x)$ contains the annihilation operators while $\phi_i^{(I-)}(x)$ contains the creation operators. Wick theorem for quantum fields can be expressed by the following formula

$$\begin{aligned} \prod_{k=1}^{m} \phi_{i_k}^{(I)}(x_k) = &:\prod_{k=1}^{m} \phi_{i_k}^{(I)}(x_k): + \sum_{(\alpha,\beta)} :\overbrace{\phi_{i_\alpha}^{(I)}(x_\alpha)\phi_{i_\beta}^{(I)}(x_\beta)} \prod_{k\neq\alpha,\beta} \phi_{i_k}^{(I)}(x_k): + \\ &+ \sum_{(\alpha,\beta),(\gamma,\delta)} :\overbrace{\phi_{i_\alpha}^{(I)}(x_\alpha)\phi_{i_\beta}^{(I)}(x_\beta)}\, \overbrace{\phi_{i_\gamma}^{(I)}(x_\gamma)\phi_{i_\delta}^{(I)}(x_\delta)} \prod_{k\neq\alpha,\beta,\gamma,\delta} \phi_{i_k}^{(I)}(x_k): + \cdots + \\ &+ \ \textit{full contracted term}. \end{aligned} \tag{22}$$

Let us elucidate this formula with same examples:

Example 1: $m = 2$ **fields**

$$\phi_{i_1}^{(I)}(x)\phi_{i_2}^{(I)}(y) =: \phi_{i_1}^{(I)}(x)\phi_{i_2}^{(I)}(y) : + \underbrace{:\overbrace{\phi_{i_1}^{(I)}(x)\phi_{i_2}^{(I)}(y)}:}_{full\ contracted\ term}; \tag{23}$$

Example 2: $m = 4$ **fields**

$$\phi_{i_1}^{(I)}(x)\phi_{i_2}^{(I)}(y)\phi_{i_3}^{(I)}(z)\phi_{i_4}^{(I)}(w) =: \phi_{i_1}^{(I)}(x)\phi_{i_2}^{(I)}(y)\phi_{i_3}^{(I)}(z)\phi_{i_4}^{(I)}(w) : +$$

$$+ : \phi_{i_1}^{(I)}\overline{(x)\phi_{i_2}^{(I)}}(y)\phi_{i_3}^{(I)}(z)\phi_{i_4}^{(I)}(w) : + : \phi_{i_1}^{(I)}\overline{(x)\phi_{i_3}^{(I)}}(z)\phi_{i_4}^{(I)}(w)\phi_{i_2}^{(I)}(y) : + : \phi_{i_1}^{(I)}\overline{(x)\phi_{i_4}^{(I)}}(w)\phi_{i_2}^{(I)}(y)\phi_{i_3}^{(I)}(z) : +$$

$$+ : \phi_{i_2}^{(I)}\overline{(y)\phi_{i_3}^{(I)}}(z)\phi_{i_1}^{(I)}(x)\phi_{i_4}^{(I)}(w) : + : \phi_{i_2}^{(I)}\overline{(y)\phi_{i_4}^{(I)}}(w)\phi_{i_1}^{(I)}(x)\phi_{i_3}^{(I)}(z) : + : \phi_{i_3}^{(I)}\overline{(z)\phi_{i_4}^{(I)}}(w)\phi_{i_1}^{(I)}(x)\phi_{i_2}^{(I)}(y) : +$$

$$+ \underbrace{: \phi_{i_1}^{(I)}\overline{(x)\phi_{i_2}^{(I)}}(y)\phi_{i_3}^{(I)}\overline{(z)\phi_{i_4}^{(I)}}(w) : + : \phi_{i_1}^{(I)}\overline{(x)\phi_{i_2}^{(I)}\overline{(y)\phi_{i_3}^{(I)}}}\overline{(z)\phi_{i_4}^{(I)}}(w) : + : \phi_{i_1}^{(I)}\overline{(x)\phi_{i_2}^{(I)}\overline{(y)\phi_{i_3}^{(I)}}(z)\phi_{i_4}^{(I)}}(w) :}_{full\ contracted\ term}. \tag{24}$$

In the case we have a time ordered product of fields, does not change much: we only have to consider the time ordering. Before moving on, let us recall the definition of time ordering.

*Definition (**time ordering of operators**): given two creation operators $\hat{A}(x)$ and $\hat{B}(y)$ we define their time ordering as*

$$\mathcal{T}\left[A(x)B(y)\right] := \begin{cases} A(x)B(y) & \text{if } x^0 > y^0, \\ \pm B(y)A(x) & \text{if } x^0 < y^0. \end{cases}; \tag{25}$$

the $\pm$ depends on if the operators are bosonic or fermionic.

In that case we would have

$$\begin{aligned} \mathcal{T}[\prod_{k=1}^{m}\phi_{i_k}^{(I)}(x_k)] = \mathcal{T}[:\prod_{k=1}^{m}\phi_{i_k}^{(I)}(x_k):] + \sum_{(\alpha,\beta)}\mathcal{T}[:\phi_{i_\alpha}^{(I)}\overline{(x_\alpha)\phi_{i_\beta}^{(I)}}(x_\beta)\prod_{k\neq\alpha,\beta}\phi_{i_k}^{(I)}(x_k):] + \\ + \sum_{(\alpha,\beta),(\gamma,\delta)}\mathcal{T}[:\phi_{i_\alpha}^{(I)}\overline{(x_\alpha)\phi_{i_\beta}^{(I)}}(x_\beta)\ \phi_{i_\gamma}^{(I)}\overline{(x_\gamma)\phi_{i_\delta}^{(I)}}(x_\delta)\prod_{k\neq\alpha,\beta,\gamma,\delta}\phi_{i_k}^{(I)}(x_k):] + \cdots + \\ + \mathcal{T}[\ full\ contracted\ term]. \end{aligned} \tag{26}$$

Let us give the simplest example that will be useful in the following

Example 1: $m = 2$ **fields**

$$\mathcal{T}[\phi_{i_1}^{(I)}(x)\phi_{i_2}^{(I)}(y)] = \mathcal{T}[:\phi_{i_1}^{(I)}(x)\phi_{i_2}^{(I)}(y):] + \mathcal{T}[:\phi_{i_1}^{(I)}\overline{(x)\phi_{i_2}^{(I)}}(y):]. \tag{27}$$

In order to conclude our digression on the Wick theorem, we wont now demonstrate that the contraction of two fields is a distribution and not an operator, therefore it can be brought in or out of the normal ordering, and its vacuum expectation value is exactly the Feynman propagator. With this goal, we take the vacuum expectation value of equation (27) we get (we will understand why the first equality holds in a while)

$$\mathcal{T}[:\overbrace{\phi^{(I)}_{i_1}(x)\phi^{(I)}_{i_2}}(y):] \stackrel{?}{=} \langle 0|\mathcal{T}[:\overbrace{\phi^{(I)}_{i_1}(x)\phi^{(I)}_{i_2}}(y):]|0\rangle = \langle 0|\mathcal{T}[\phi^{(I)}_{i_1}(x)\phi^{(I)}_{i_2}(y)]|0\rangle - \underbrace{\langle 0|\mathcal{T}[:\phi^{(I)}_{i_1}(x)\phi^{(I)}_{i_2}(y):]|0\rangle}_{=0}. \tag{28}$$

where we are assuming a normalized vacuum state $|0\rangle$ and we used that vacuum expectation values of any normal ordering is identically zero. We can see that (using bold symbols for threevectors)

$$\begin{aligned}
&\langle 0|\mathcal{T}[\phi^{(I)}_{i_1}(x)\phi^{(I)}_{i_2}(y)]|0\rangle = \\
&= \int \frac{d^3\mathbf{p}\; d^3\mathbf{p'}}{(2\pi)^3\sqrt{4\omega_{\mathbf{p}}\omega_{\mathbf{p'}}}} \sum_s \sum_{s'} \langle 0|\hat{a}^{(s)}(\mathbf{p})\hat{a}^{(s')\dagger}(\mathbf{p'})|0\rangle \times \\
&\times \left[\Theta(x^0-y^0)\epsilon^{(s)}_{i_1}\epsilon^{*(s')}_{i_2} e^{i(-p\cdot x+p'\cdot y)} + \Theta(y^0-x^0)\epsilon^{(s)}_{i_2}\epsilon^{(*s')}_{i_1} e^{i(-p\cdot y+p'\cdot x)}\right] = \\
&= \int \frac{d^3\mathbf{p}\; d^3\mathbf{p'}}{(2\pi)^3\sqrt{4\omega_{\mathbf{p}}\omega_{\mathbf{p'}}}} \sum_s \sum_{s'} \delta(\mathbf{p}-\mathbf{p'})\delta_{ss'} \times \\
&\times \left[\Theta(x^0-y^0)\epsilon^{(s)}_{i_1}\epsilon^{*(s')}_{i_2} e^{i(-p\cdot x+p'\cdot y)} + \Theta(y^0-x^0)\epsilon^{(s)}_{i_2}\epsilon^{*(s')}_{i_1} e^{i(-p\cdot y+p'\cdot x)}\right] = \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2\omega_{\mathbf{p}}} \sum_s \left[\Theta(x^0-y^0)\epsilon^{(s)}_{i_1}\epsilon^{*(s)}_{i_2} e^{-i(p\cdot(x-y)} + \Theta(y^0-x^0)\epsilon^{(s)}_{i_2}\epsilon^{*(s)}_{i_1} e^{ip\cdot(x-y)}\right] \equiv iG_F(x-y).
\end{aligned} \tag{29}$$

where the sum over s means the sum over polarisations.
The vacuum expectation value of the time ordered product of two fields is equal to the time ordered product of the full contracted term of two fields and both are equal to the Feynman propagator. Therefore, the time ordered product of the full contracted term is not an operator but a distribution given by the product of Feynman propagators. This means two things: first it can be brought out or in the normal ordering at will and, second, it does not act on states and can be brought out or in expectation values.

We can now really appreciate the power of Wick theorem. If we want to compute the vacuum expectation value of a time ordered product of fields (and so an m-point correlation function) we can use (26) to rewrite it as

$$\langle 0|\mathcal{T}[\prod_{k=1}^{m}\phi^{(I)}_{i_k}(x_k)]|0\rangle = \langle 0|\textit{full contracted term}|0\rangle = \sum_{l=1}^{(m-1)!!}\left[\prod_{j=1}^{\frac{m}{2}} iG^{(l)}_{F_j}(x^{(j)}_1 - x^{(j)}_2)\right]; \tag{30}$$

this is because, as we learned, the vacuum expectation value of a normal ordering is identically zero and since non-fully contracted terms contain normal orders they do not matter. Moreover, we see that the correlation function of an odd number of fields is identically vanishing simple because there is no possibility to get a fully contracted term. The meaning of equation (30) is that we need to sum $(m-1)!! := \frac{m!}{2^{\frac{m}{2}}\frac{m}{2}!}$ terms each of which given by the product of $\frac{m}{2}$ Feynman propagators each of which computed at a couple of different points. Let us elucidate with some examples.

Example 1 $m = 2$ **fields**

For two fields we have $\frac{m}{2} = 1, (m-1)!! = 1$; therefore

$$\langle 0|\mathcal{T}[\phi^{(I)}_{i_1}(x)\phi^{(I)}_{i_2}(y)]|0\rangle = iG_F(x-y). \tag{31}$$

Example 2 $m = 4$ **fields**

For four fields we have $\frac{m}{2} = 2, (m-1)!! = 3$; therefore

$$\begin{aligned}\langle 0|\mathcal{T}[\phi^{(I)}_{i_1}(x)\phi^{(I)}_{i_2}(y)\phi^{(I)}_{i_3}(z)\phi^{(I)}_{i_4}(w)]|0\rangle = \\ &= i^2[G_F(x-y)G_F(z-w)+ \\ &+ G_F(x-z)G_F(y-w)+ \\ &+ G_F(x-w)G_F(y-z)].\end{aligned} \tag{32}$$

4 The Isserling theorem

As true in general, physicists find results that mathematicians already knew twenty/fifty years earlier. The case of the Wick theorem is no an exception. Indeed the work of Wick is from about 1950 while the Isserling theorem, an identical theorem from gaussian distribution in probability theory, is from about 1920. In this section we want to stress that Wick theorem is nothing but Isserling theorem. First of all let us enunciate the theorem.

*Theorem (**Isserling**): If $(X_1, \ldots, X_m)$ is a zero-mean multivariate normal random vector, then*

$$\mathrm{E}[\,X_1X_2\cdots X_m\,] = \sum_{p\in P^2_m}\prod_{\{i,j\}\in p}\mathrm{E}[\,X_iX_j\,],; \tag{33}$$

where the sum is over all the pairings of $\{1,\ldots,m\}$, namely all distinct ways of partitioning $\{1,\ldots,m\}$ into pairs $\{i,j\}$, and the product is over the pairs contained in p.

We note that is m is odd there is no way have a pairing, therefore $\mathrm{E}[\,X_1X_2\cdots X_m\,] = 0$, while if m is even there are exactly $(m-1)!!$ pairings. This is exactly what happens using Wick theorem on quantum fields after compiling the following dictionary:

$$\begin{aligned}&\mathrm{E}[\,X_1X_2\cdots X_m\,] \to \langle 0|\mathcal{T}[\prod_{k=1}^{m}\phi^{(I)}_{i_k}(x_k)]|0\rangle; \\ &\mathrm{E}[\,X_iX_j\,] \to \langle 0|\mathcal{T}[\phi^{(I)}_{i_1}(x)\phi^{(I)}_{i_2}(y)]|0\rangle = iG_F(x-y).\end{aligned} \tag{34}$$

At first look seems only a pure coincidence: there is no reason why a set of fields has to behave like a zero-mean multivariate normal random vector. Anyway this is no true. In path integral formulation of QFT, all the dynamics is encoded in the generating functional Z. This is the analogous of the partition function in statistical systems and taking functional derivatives of Z we can evaluate correlation functions. The general expression of the generating functional and the n-point correlation functions are

$$Z := \int \mathcal{D}\phi_{i_k}e^{iS[\phi_{i_k}]}, \qquad G_n(x_1,...,x_n) := \int \mathcal{D}\phi_{i_k}\phi_{i_k}(x_1)...\phi_{i_k}(x_n)e^{iS[\phi_{i_k}]}; \tag{35}$$

where $\mathcal{D}\phi_{i_k}$ is an integration measure over the space of fields. Without entering in path integral formulation, the crucial point is that every free theory is expressed by an action whose Lagrangian is quadratic in the fields. Therefore, upon integration by parts every free theory action can be written in the schematic form

$$S[\phi_{i_k}] = \int d^4x \frac{1}{2}\phi_{i_k}(x)[D]\phi_{i_k}(x); \tag{36}$$

for example for a single Klein-Gordon field we have $D = \Box + m^2$. Inserting in the definitions of generating functional and of n-point correlation functions we have

$$Z = \int \mathcal{D}\phi_{i_k} e^{i\int d^n x \frac{1}{2}\phi_{i_k}(x)[D]\phi_{i_k}(x)}, \;\; G_n(x_1, ..., x_n) = \int \mathcal{D}\phi_{i_k}\phi_{i_k}(x_1)...\phi_{i_k}(x_n) e^{i\int d^n x \frac{1}{2}\phi_{i_k}(x)[D]\phi_{i_k}(x)}. \tag{37}$$

We may note that Z is nothing but a functional normal integral, namely the infinite dimensional case of the standard normal integral

$$g = \int e^{-\left(\frac{1}{2}x_l A_{ls} x_s\right)} \, d^n x \tag{38}$$

where the integral over x is replaced by a functional integral over ϕ_{i_k} and the sums over indexes is replaced by integration over x, while $G_n(x_1, ..., x_n)$ are nothing but the expectation value of the vector $(\phi_{i_k}(x_1), ..., \phi_{i_k}(x_n))$ with functional distribution given by a functional normal, namely they are the analogue of

$$\mathrm{E}[\, X_1 X_2 \cdots X_n \,] = \int x_1...x_n e^{-\left(\frac{1}{2}x_l A_{ls} x_s\right)} \, d^n x, \tag{39}$$

indeed n-point correlation function are expectation values.

I this perspective we note that Isserling theorem is Wick theorem applied to functional normal distribution of fields; therefore one may think, at least a the free theory level, QFT as nothing but a functional probability theory. It would be interesting understand if some of these considerations are still true for interacting theory. Unlikely, the action for interacting theory in no longer quadratic and the functional distribution is no longer a normal one; moreover, there is no an obvious analogue of Isserling theorem for non.normal distribution in probability theory. In probability theory there exist a way to compute the expectation value for non-Gaussian random variables in terms of joint cumulants, but at the moment, with the best of our knowledge, there is no an analogue in QFT. Anyway, this is an important goal since if one find an analogue of Wick theorem for non-normal (so interacting fields), one could compute correlation functions of the interacting theory (so non-normal distributed fields) without perturbative expansion in free theory correlation functions. This would be a full non-perturbative way to compute interacting correlation functions.